Math 105, Fall 2013

Worksheet 7, Tuesday, October 29, 2013, Answer Key

1. Suppose that f and g are functions, **and**

- $\lim_{x\to 5} f(x) = 6$
- $g(3) = 5$
- $\lim_{x\to 7} g(x) = 1$
- $g(x)$ is continuous at $x = 3$.
- $f(x)$ is continuous at $x = 5$
- $g(7) = 9$

Evaluate the following quantities and fully **justify** your answers. Do not just put down a value:

(a) $\lim_{x\to 3} g(x) = g(3) = \boxed{5}$ The first equality holds because of the assumption of g being continuous at $x = 3$. The second equality was given in the assumptions.

(b) $f(5) = \lim_{x\to 5} f(x) = \boxed{6}$
The first equality holds because of the assumption of f being continuous at $x = 5$. The second equality was given in the assumptions.

(c) Compute $f \circ g(3) = f(g(3)) = f(5) = \boxed{6}$ using the given info $g(3) = 5$ and answer from (b) above.

(d) Is $g(x)$ continuous at $x = 7$? Why or why not? Use math notation.
$g(x)$ is $\boxed{\text{not continuous}}$ because $9 = g(7) \neq \lim_{x\to 7} g(x) = 1$

2. Compute the derivative of each of the following functions. For these problems, you do **not** need to simplify your derivative.

(a) $$y = \frac{1}{\left(\frac{1}{x^7} + \sqrt{x^6 - 7}\right)^{\frac{6}{7}}} = \left(\frac{1}{x^7} + \sqrt{x^6 - 7}\right)^{-\frac{6}{7}}$$

$$y' = \boxed{-\frac{6}{7}\left(\frac{1}{x^7} + \sqrt{x^6 - 7}\right)^{-\frac{13}{7}}\left(\frac{-7}{x^8} + \frac{1}{2\sqrt{x^6 - 7}}(6x^5)\right)}$$

(b) $$y = \sqrt{x} + \frac{1}{\sqrt{x}} + \frac{1}{1 + \sqrt{x}} + \frac{1}{\sqrt{1 + x}} = \sqrt{x} + x^{-\frac{1}{2}} + \left(1 + \sqrt{x}\right)^{-1} + (1 + x)^{-\frac{1}{2}}$$

$$y' = \boxed{\frac{1}{2\sqrt{x}} - \frac{1}{2}x^{-\frac{3}{2}} - \left(1 + \sqrt{x}\right)^{-2}\left(\frac{1}{2\sqrt{x}}\right) - \frac{1}{2}(1 + x)^{-\frac{3}{2}}}$$

(c) $f(x) = \left(x^4 - \frac{4}{x^4}\right)\left(4x + \frac{\sqrt{x}}{4}\right)$

$$f'(x) = \boxed{\left(x^4 - \frac{4}{x^4}\right)\left(4 + \left(\frac{1}{4}\right)\frac{1}{2\sqrt{x}}\right) + \left(4x + \frac{\sqrt{x}}{4}\right)\left(4x^3 + \frac{16}{x^5}\right)}$$

it's not necessary, but we can simplify the function to match it the other method

$$f'(x) = 4x^4 + \frac{x^{\frac{7}{2}}}{8} - \frac{16}{x^4} - \frac{1}{2x^{\frac{9}{2}}} + 16x^4 + \frac{64}{x^4} + x^{\frac{7}{2}} + \frac{4}{x^{\frac{9}{2}}} = 20x^4 + \frac{9}{8}x^{\frac{7}{2}} + \frac{48}{x^4} + \frac{7}{2x^{\frac{9}{2}}}$$

OR you can simplify the function first and then use the power rules...

$$f(x) = \left(x^4 - \frac{4}{x^4}\right)\left(4x + \frac{\sqrt{x}}{4}\right) = 4x^5 + \frac{1}{4}x^{\frac{9}{2}} - \frac{16}{x^3} - \frac{1}{x^{\frac{7}{2}}}$$

$$f'(x) = \boxed{20x^4 + \frac{9}{8}x^{\frac{7}{2}} + \frac{48}{x^4} + \frac{7}{2}\frac{1}{x^{\frac{5}{2}}}}$$

(d) $y = \left(\frac{1}{3x^8} + 8x^3\right)^{\frac{3}{8}}$

$$y' = \boxed{\frac{3}{8}\left(\frac{1}{3x^8} + 8x^3\right)^{-\frac{5}{8}}\left(-\frac{8}{3x^9} + 24x^2\right)}$$

(e) $w(t) = \frac{(3t+1)^4}{(4t+1)^3}$

$$w'(t) = \boxed{\frac{(4t+1)^3 4(3t+1)^3(3) - (3t+1)^4(3)(4t+1)^2(4)}{(4t+1)^6}}$$

3. Find the equation of the tangent line to this curve $y = \frac{6x}{\sqrt{x^2+3}}$ at the point where $x = 1$.

$$y' = \frac{\sqrt{x^2+3}\,(6) - 6x\left(\frac{1}{2\sqrt{x^2+3}}(2x)\right)}{x^2+3}$$

You can simplify the derivative if you want, but you should go straight to plugging in $x = 1$ and simplifying the value of the specific slope. Easier.

$$y'(1) = \frac{\sqrt{1+3}\,(6) - 6\left(\frac{1}{2\sqrt{1+3}}(2)\right)}{1+3} = \frac{2(6) - \frac{6}{2}}{4}$$

$$= \frac{12-3}{4} = \frac{9}{4}$$

The point is $(1, y(1)) = (1, 3)$.

The equation of the tangent line through the point $(1, 3)$ with slope $\frac{9}{4}$ is given by

$y - 3 = \frac{9}{4}(x-1)$

or $y - 3 = \frac{9}{4}x - \frac{9}{4}$

or $y = \frac{9}{4}x - \frac{9}{4} + 3$

or $y = \frac{9}{4}x - \frac{9}{4} + \frac{12}{4}$

or $y = \boxed{\frac{9}{4}x + \frac{3}{4}}$

4. Compute the derivative of $f(x) = \dfrac{6-5x}{2+4x}$ **two** different ways:
   - First use the **limit definition of the derivative**.
   - Second use the Quotient Rule.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{6-5(x+h)}{2+4(x+h)} - \frac{6-5x}{2+4x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{[6-5x-5h](2+4x)-(6-5x)[2+4x+4h]}{(2+4(x+h))(2+4x)}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{12-10x-10h+24x-20x^2-20xh-(12+24x+24h-10x-20x^2-20xh)}{(2+4(x+h))(2+4x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{12-10x-10h+24x-20x^2-20xh-12-24x-24h+10x+20x^2+20xh}{(2+4(x+h))(2+4x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{-34h}{(2+4(x+h))(2+4x)}\right)}{h} = \lim_{h\to 0} \frac{-34h}{(2+4(x+h))(2+4x)}\left(\frac{1}{h}\right)$$

$$= \lim_{h\to 0} \frac{-34}{(2+4(x+h))(2+4x)} = \boxed{\frac{-34}{(2+4x)^2}}$$

Other method, Quotient Rule:

$$f'(x) = \frac{(2+4x)(-5)-(6-5x)(4)}{(2+4x)^2} = \frac{-10-20x-24+20x}{(2+4x)^2} = \boxed{\frac{-34}{(2+4x)^2}} \text{ match!}$$

Next, simplify your answer in the first part. Then compute the second derivative $f''(x)$.

$f(x) = (-34)(2+4x)^{-2}$

$$f''(x) = (-34)(-2)(2+4x)^{-3}(4) = \boxed{\frac{272}{(2+4x)^3}}$$

5. Find **all** x-coordinates at which the graphs of the following functions have horizontal tangent lines. Please **simplify** your derivatives first. Why?

(a) $f(x) = (4x+1)^4(7-3x)^8$

$$\begin{aligned} f'(x) &= (4x+1)^4 8(7-3x)^7(-3) + (7-3x)^8 4(4x+1)^3(4) \\ &= 4(4x+1)^3(7-3x)^7\left[(-6)(4x+1) + 4(7-3x)\right] \\ &= 4(4x+1)^3(7-3x)^7\left[-24x - 6 + 28 - 12x\right] \\ &= 4(4x+1)^3(7-3x)^7\left[-36x + 22\right] \overset{\text{set}}{=} 0 \end{aligned}$$

This means that

$4x+1=0$ or $7-3x=0$ or $-36x+22=0$

which implies

$\boxed{x=-\frac{1}{4}}$ or $\boxed{x=\frac{7}{3}}$ or $\boxed{x=\frac{11}{18}}$

(b) $G(t) = t^3(1-t)^7$

$$\begin{aligned} G'(t) &= t^3 7(1-t)^6(-1) + (1-t)^7(3t^2) \\ &= t^2(1-t)^6\left[-7t + 3(1-t)\right] \\ &= t^2(1-t)^6\left[-10t+3\right] \end{aligned}$$

This means that

$t=0$ or $1-t=0$ or $-10t+3=0$

which implies

$\boxed{t=0}$ or $\boxed{t=1}$ or $\boxed{t=\frac{3}{10}}$

6. Compute the derivative of $f(x) = \dfrac{x}{x-1} + \dfrac{x}{x+1}$. Simplify your answer to a single fraction.

We have two options. You can first simplify the function to a single fraction and then differentiate. Or you can differentiate first and then simplify.

First let's simplify $f(x) = \dfrac{x}{x-1} + \dfrac{x}{x+1} = \left(\dfrac{x+1}{x+1}\right)\dfrac{x}{x-1} + \dfrac{x}{x+1}\left(\dfrac{x-1}{x-1}\right) = \dfrac{x(x+1)+x(x-1)}{(x-1)(x+1)} = \dfrac{x^2+x+x^2-x}{(x-1)(x+1)} = \dfrac{2x^2}{x^2-1}$.

Now let's differentiate

$$f'(x) = \frac{(x^2-1)(4x) - 2x^2(2x)}{(x^2-1)^2} = \frac{4x^3 - 4x - 4x^3}{(x^2-1)^2} = \boxed{\frac{-4x}{(x^2-1)^2}}$$

OR let's differentiate first...and then find a common denominator

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} + \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{-1}{(x-1)^2} + \frac{1}{(x+1)^2}$$

$$= \frac{-(x+1)^2 + (x-1)^2}{(x-1)^2(x+1)^2} = \frac{-(x^2+2x+1) + x^2 - 2x + 1}{(x-1)^2(x+1)^2}$$

$$= \frac{-x^2 - 2x - 1 + x^2 - 2x + 1}{(x-1)^2(x+1)^2} = \frac{-4x}{(x-1)^2(x+1)^2} = \boxed{\frac{-4x}{(x^2-1)^2}}$$

7. Find the equation of the tangent line to the curve $x^3 + x^2y = 6 - 4y^2$ at the point $(1, 1)$.

First, implicitly differentiate by taking the derivative of both sides:

$$\frac{d}{dx}\left(x^3 + x^2y\right) = \frac{d}{dx}\left(6 - 4y^2\right)$$

$$3x^2 + x^2\frac{dy}{dx} + y(2x) = 0 - 8y\frac{dy}{dx}$$

We can plug the point $(1, 1)$ immediately and solve for the derivative value $\frac{dy}{dx}$

$$3(1)^2 + (1)^2\frac{dy}{dx} + (1)(2(1)) = 0 - 8(1)\frac{dy}{dx}$$

$$3 + \frac{dy}{dx} + 2 = -8\frac{dy}{dx}$$

Isolate and solve for $\frac{dy}{dx}$

$$9\frac{dy}{dx} = -5$$

Finally, $\frac{dy}{dx} = -\frac{5}{9}$.

Now the equation of the tangent line at the point $(1, 1)$ with slope $-\frac{5}{9}$ is given by

$$y - 1 = -\frac{5}{9}(x - 1)$$

or $y - 1 = -\frac{5}{9}x - \frac{5}{9}$

or $y = -\frac{5}{9}x - \frac{5}{9} + 1$

or $y = -\frac{5}{9}x + \frac{5}{9} + \frac{9}{9}$

or $y = \boxed{-\frac{5}{9}x + \frac{14}{9}}$