Math 105, Fall 2013

Worksheet 6, Tuesday, October 22, 2013, Answer Key

1. Prove that $y = x^{\frac{2}{3}}$ is not differentiable at $x = 0$.

   We need to show that $f'(0)$ does not exist.

   $$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^{\frac{2}{3}} - 0}{h} = \lim_{h\to 0} \frac{h^{\frac{2}{3}}}{h} = \lim_{h\to 0} \frac{1}{h^{\frac{1}{3}}}$$

   $\boxed{\text{Does Not Exist b/c RHL} \neq \text{LHL}}$

   RHL: $= \lim_{h\to 0} \frac{1}{h^{\frac{1}{3}}} = \frac{1}{0^+} = +\infty$

   LHL: $\lim_{h\to 0^-} \frac{1}{h^{\frac{1}{3}}} = \frac{1}{0^-} = -\infty$

2. Compute the derivative of each of the following functions. For these problems, you do **not** need to simplify your derivative.

   (a) $y = \sqrt{x} = x^{\frac{1}{2}}$

   $y' = \frac{1}{2}x^{-\frac{1}{2}} = \boxed{\frac{1}{2\sqrt{x}}}$

   (b) $y = \left(\sqrt{x} + \frac{1}{x}\right)^9$

   $y' = \boxed{9\left(\sqrt{x} + \frac{1}{x}\right)^8 \left[\frac{1}{2\sqrt{x}} - \frac{1}{x^2}\right]}$

   (c) $f(x) = \left(x^2 - \frac{5}{x^2}\right)(3x + \sqrt{x})$

   $f'(x) = \boxed{\left(x^2 - \frac{5}{x^2}\right)\left(3 + \frac{1}{2\sqrt{x}}\right) + (3x + \sqrt{x})\left(2x + \frac{10}{x^3}\right)}$

   (d) $y = \frac{1}{\sqrt{x^2 - 5x + 3}} = \left(x^2 - 5x + 3\right)^{-\frac{1}{2}}$

   $y' = \boxed{-\frac{1}{2}\left(x^2 - 5x + 3\right)^{-\frac{3}{2}}(2x - 5)}$

   (e) $y = \left(\frac{1}{x^3} + 7x\right)^{\frac{5}{7}} \left(x^4 - \frac{1}{x^7}\right)^{-5}$

   $y' = \boxed{\left(\frac{1}{x^3} + 7x\right)^{\frac{5}{7}} (-5)\left(x^4 - \frac{1}{x^7}\right)^{-6} (4x^3 + 7x^{-8})}$ continued...

   $\boxed{+\left(x^4 - \frac{1}{x^7}\right)^{-5} \left(\frac{5}{7}\right)\left(\frac{1}{x^3} + 7x\right)^{-\frac{2}{7}} \left(-\frac{3}{x^4} + 7\right)}$

(f) $y = \sqrt{\dfrac{x+5}{5-x}}$

$$y' = \boxed{\frac{1}{2\sqrt{\frac{x+5}{5-x}}}\left(\frac{(5-x)(1)-(x+5)(-1)}{(5-x)^2}\right)}$$

3. Find the equation of the tangent line to this curve $y = \sqrt{x + (x^2+1)^3}$ at the point where $x = 1$.

First we need slope. Compute the derivative.

$$y' = \frac{1}{2\sqrt{x+(x^2+1)^3}}\left(1 + 3(x^2+1)^2(2x)\right)$$

The specific slope at $x = 1$ is $y'(1) = \dfrac{1}{2\sqrt{1+(2)^3}}\left(1+3(2)^2(2)\right) = \dfrac{1}{2\sqrt{9}}(25) = \dfrac{25}{6}$

The point is $(1, y(1)) = (1, \sqrt{1+2^3}) = (1, \sqrt{9}) = (1, 3)$

Using point-slope form, we find the equation of the tangent line

$$y - 3 = \frac{25}{6}(x-1)$$

or

$$y - 3 = \frac{25}{6}x - \frac{25}{6}$$

or finally

$$y = \frac{25}{6}x - \frac{25}{6} + 3$$

$$y = \frac{25}{6}x - \frac{25}{6} + \frac{18}{6}$$

$$\boxed{y = \frac{25}{6}x - \frac{7}{6}}$$

4. Compute the derivative of $f(x) = \sqrt{x^3 - 4x^2 + \dfrac{x}{3} - 7}$ **two** different ways:

- First use the **limit definition of the derivative**.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{(x+h)^3 - 4(x+h)^2 + \frac{x+h}{3} - 7} - \sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{(x+h)^3 - 4(x+h)^2 + \frac{x+h}{3} - 7} - \sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}}{h} \text{ conjugate...}$$

$$\cdot \left(\frac{\sqrt{(x+h)^3 - 4(x+h)^2 + \frac{x+h}{3} - 7} + \sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}}{\sqrt{(x+h)^3 - 4(x+h)^2 + \frac{x+h}{3} - 7} + \sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}} \right)$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - 4(x+h)^2 + \frac{x+h}{3} - 7 - \left(x^3 - 4x^2 + \frac{x}{3} - 7\right)}{h\left(\sqrt{(x+h)^3 - 4(x+h)^2 + \frac{x+h}{3} - 7} + \sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}\right)}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 4x^2 - 8xh - 4h^2 + \frac{x}{3} + \frac{h}{3} - 7 - x^3 + 4x^2 - \frac{x}{3} + 7}{h\left(\sqrt{(x+h)^3 - 4(x+h)^2 + \frac{x+h}{3} - 7} + \sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}\right)}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 8xh - 4h^2 + \frac{h}{3}}{h\left(\sqrt{(x+h)^3 - 4(x+h)^2 + \frac{x+h}{3} - 7} + \sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}\right)}$$

$$= \lim_{h\to 0} \frac{h\left(3x^2 + 3xh + h^2 - 8x - 4h + \frac{1}{3}\right)}{h\left(\sqrt{(x+h)^3 - 4(x+h)^2 + \frac{x+h}{3} - 7} + \sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}\right)}$$

$$= \lim_{h\to 0} \frac{3x^2 + 3xh + h^2 - 8x - 4h + \frac{1}{3}}{\sqrt{(x+h)^3 - 4(x+h)^2 + \frac{x+h}{3} - 7} + \sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}}$$

$$= \boxed{\frac{3x^2 - 8x + \frac{1}{3}}{2\sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}}}$$

• Second use the Chain Rule.

$$f(x) = \left(x^3 - 4x^2 + \frac{x}{3} - 7\right)^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}\left(x^3 - 4x^2 + \frac{x}{3} - 7\right)^{-\frac{1}{2}}\left(3x^2 - 8x + \frac{1}{3}\right) = \boxed{\frac{3x^2 - 8x + \frac{1}{3}}{2\sqrt{x^3 - 4x^2 + \frac{x}{3} - 7}}}$$

5. Simplify the expression $3(x+1)^2(1-2x)^4+(x+1)^3 4(1-2x)^3(-2)$

   Hint: Common factors.

   $3(x+1)^2(1-2x)^4+(x+1)^3 4(1-2x)^3(-2)$

   $= (x+1)^2(1-2x)^3\left[3(1-2x)-8(x+1)\right]$

   $= (x+1)^2(1-2x)^3\left[3-6x-8x-8\right]$

   $= \boxed{(x+1)^2(1-2x)^3\left[-14x-5\right]}$

6. Solve the equation $\dfrac{2x+1}{\sqrt{x}}-\dfrac{x\sqrt{x}}{x+2}=0$

   Hint: First, put the left hand side over a common denominator. Then remember that a fraction is zero when the numerator is zero.

   $$\frac{(2x+1)(x+2)}{\sqrt{x}(x+2)}-\frac{(x\sqrt{x})\sqrt{x}}{\sqrt{x}(x+2)}=0$$

   $$\frac{(2x+1)(x+2)-x^2}{\sqrt{x}(x+2)}=0$$

   $$\frac{2x^2+5x+2-x^2}{\sqrt{x}(x+2)}=0$$

   $$\frac{x^2+5x+2}{\sqrt{x}(x+2)}=0$$

   This implies the numerator $x^2+5x+2=0$ by clearing the denominator.

   This doesn't factor obviously, so we use the quadratic formula.

   $$x=\frac{-5\pm\sqrt{25-8}}{2}=\boxed{\frac{-5\pm\sqrt{17}}{2}}$$

7. For later purposes we need to practice solving.

   (a) Consider the equation $x^2+2xyy'=3y-7y'$. Solve for y'.
   $2xyy'+7y'=3y-x^2$
   $y'(2xy+7)=3y-x^2$
   $y'=\boxed{\dfrac{3y-x^2}{2xy+7}}$

(b) Consider the equation $3y^2\frac{dy}{dx} - 5x^3y = 4x + 7\frac{dy}{dx}$. Solve for $\frac{dy}{dx}$.

$3y^2\frac{dy}{dx} - 7\frac{dy}{dx} = 4x + 5x^3y$

$\frac{dy}{dx}(3y^2 - 7) = 4x + 5x^3y$

Finally, $\frac{dy}{dx} = \boxed{\frac{4x + 5x^3y}{3y^2 - 7}}$

8. Find **all** x-coordinates at which the graphs of the following functions have horizontal tangent lines. Please **simplify** your derivatives first. Why?

(a) $f(x) = (7x - 3)^4(5x + 2)^6$

$f'(x) = (7x - 3)^4 6(5x + 2)^5(5) + (5x + 2)^6 4(7x - 3)^3(7)$

$= (7x - 3)^3 2(5x + 2)^5\left((7x - 3)(15) + (5x + 2)(14)\right)$

$= (7x - 3)^3 2(5x + 2)^5\left(105x - 45 + 70x + 28\right)$

$= (7x - 3)^3 2(5x + 2)^5\left(175x - 17\right) \stackrel{\text{set}}{=} 0$

For a product of several terms to equal 0, then any of the terms could be equal to 0. Therefore,

$7x - 3 = 0$ OR $5x + 2 = 0$ OR $175x - 17 = 0$

Finally, solving each equation

$\boxed{x = \frac{3}{7}}$ OR $\boxed{x = -\frac{2}{5}}$ OR $\boxed{x = \frac{17}{175}}$

(b) $f(x) = (x + 1)^2\sqrt{x + 2}$

$f'(x) = (x + 1)^2\frac{1}{2\sqrt{x + 2}} + \sqrt{x + 2}(2)(x + 1)$

$= (x + 1)^2\frac{1}{2\sqrt{x + 2}} + \sqrt{x + 2}(2)(x + 1)\cdot\left(\frac{2\sqrt{x + 2}}{2\sqrt{x + 2}}\right)$ common denominator

$= \frac{(x + 1)^2}{2\sqrt{x + 2}} + \frac{(x + 2)(4)(x + 1)}{2\sqrt{x + 2}}$

$= \frac{(x + 1)^2 + 4(x + 2)(x + 1)}{2\sqrt{x + 2}} = \frac{x^2 + 2x + 1 + 4x^2 + 12x + 8}{2\sqrt{x + 2}}$

$= \frac{5x^2 + 14x + 9}{2\sqrt{x + 2}} \stackrel{\text{set}}{=} 0$

Therefore, clearing the denominator $5x^2 + 14x + 9 = 0$.

Factor, $5x^2 + 14x + 9 = (5x + 9)(x + 1) = 0$. So $\boxed{x = -\frac{9}{5}}$ OR $\boxed{x = -1}$

(c) $w(t) = t^2(1-t)^6$

$w'(t) = t^2 6(1-t)^5(-1) + (1-t)^6(2t)$

$= 2t(1-t)^5(-3t + (1-t))$

$= 2t(1-t)^5(-4t+1) \overset{\text{set}}{=} 0$

That implies $t = 0$ OR $1 - t = 0$ OR $-4t + 1 = 0$

so that $\boxed{t = 0}$ OR $\boxed{t = 1}$ OR $\boxed{t = \frac{1}{4}}$

9. Let $f(x)$ and $g(x)$ be differentiable functions with the following table of values:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	4	-3	2	7
2	-2	6	1	5
3	3	-2	-1	0

Let

$$h(x) = f(x) \cdot g(x)$$

$$k(x) = \frac{g(x)}{f(x)}$$

$$P(x) = f(x) \cdot f(x)$$

$$Q(x) = f \circ g(x)$$

$$W(x) = g \circ g(x).$$

Compute $h'(1)$, $k'(3)$, $P'(1)$, $Q'(2)$, and $W'(1)$.

$h'(1) = f(1)g'(1) + g(1)f'(1) = (4)(7) + (2)(-3) = 28 - 6 = \boxed{22}$

$$k'(3) = \frac{f(3)g'(3) - g(3)f'(3)}{(f(3))^2} = \frac{(3)(0) - (-1)(-2)}{(3)^2} = \frac{-2}{9} = \boxed{-\frac{2}{9}}$$

$P'(1) = f(1)f'(1) + f(1)f'(1) = 2f(1)f'(1) = 2(4)(-3) = \boxed{-24}$

OR you could rewrite $P(x) = (f(x))^2$ and use the Chain Rule. $P'(x) = 2f(x) \cdot f'(x)$.

$Q'(2) = f'(g(2))g'(2) = f'(1)(5) = (-3)(5) = \boxed{-15}$

$W'(1) = g'(g(1))g'(1) = g'(2)(7) = (5)(7) = \boxed{35}$

Note: this problem is testing whether you know your differentiation rules, especially in the case when you don't know the actual function's ($f(x)$ or $g(x)$) formula. To compute the derivative at one specific x-value, you just need the derivative information of each function piece *at* that specific x-value. You don't need to know the entire function's formula. Think about which derivative values are required in each problem. Write out the derivative carefully, and then plug in your specific x-value.