Math 105, Fall 2013

Worksheet 5, Tuesday, October 8, 2013

1. State the definition for a function $g(x)$ that is continuous at $x = -7$.

   $g(x)$ is continuous at $x = -7$ means by definition that $\boxed{\lim_{x \to -7} g(x) = g(-7)}$

2. Consider the function $f(x)$ that is continuous at $x = 3$. Assume that $f(3) = 4$.

   (a) Write the *definition* for $f(x)$ being continuous at $x = 3$.

   $f(x)$ is continuous at $x = 3$ means by definition that $\boxed{\lim_{x \to 3} f(x) = f(3)}$

   (b) Discuss what you know about $\lim_{x \to 3} f(x) = ??$ Why? Be clear and justify with mathematical notation.

   Since f is continuous at $x = 3$, we know from part (a) that $\lim_{x \to 3} f(x) = f(3)$. But we also know from the given info that $f(3) = 4$. That implies $\lim_{x \to 3} f(x) = f(3) = \boxed{4}$.

3. Suppose that f and g are functions, **and**

   - $\lim_{x \to 3} f(x) = 9$
   - $\lim_{x \to 7} g(x) = -6$
   - $\lim_{x \to 4} f(x) = 7$
   - $g(x)$ is continuous at $x = 7$.
   - $f(x)$ is continuous at $x = 4$.

   Evaluate the following quantities and fully **justify** your answers. Do not just put down a value:

   (a) $f(4) = \lim_{x \to 4} f(x) = \boxed{7}$ The first equality holds because of the assumption of f being continuous at $x = 4$. The second equality was given in the assumptions.

   (b) $g(7) = \lim_{x \to 7} g(x) = \boxed{-6}$ The first equality holds because of the assumption of g being continuous at $x = 7$. The second equality was given in the assumptions.

   (c) Compute $g \circ f(4) = g(f(4)) = g(7) = \boxed{-6}$ using part (a) and part (b).

   (d) Does $f(3) = 9$? Why or why not? Use math notation.

   We $\boxed{\text{don't know}}$ if $f(3) = 9$ because we were **not** told whether f is continuous at $x = 3$. IF f was assumed to be continuous at $x = 3$, then we would know that $f(3) = \lim_{x \to 3} f(x) = \boxed{9}$.

4. Suppose that f and g are functions, **and**

- $\lim_{x\to 7} g(x) = 3$
- $\lim_{x\to 2} g(x) = 6$
- $f(3) = 2$
- $g(x)$ is continuous at $x = 7$ and $x = 2$.
- $\lim_{x\to 3} f(x) = 5$

Evaluate the following quantities and fully **justify** your answers. Do not just put down a value:

(a) $g(7) = \lim_{x\to 7} g(x) = \boxed{3}$ The first equality holds because of the assumption of g being continuous at $x = 7$. The second equality was given in the assumptions.

(b) Compute $g \circ f(3) = g(f(3)) = g(2) = \lim_{x\to 2} g(x) = \boxed{6}$ The second equality holds because $f(3) = 2$ was given in the assumptions. The third equality holds because of the assumption that of g being continuous at $x = 2$. The last equality was given in the assumptions.

(c) Compute $f \circ g(7) = f(g(7)) = f(3) = 2$ The first equality holds from part (a). The last equality was given in the assumptions.

(d) Is $f(x)$ continuous at $x = 3$? Why or why not? Use math notation.
$\boxed{\text{No}}$ $f(x)$ is not continuous at $x = 3$ because $f(3) = \boxed{2} \neq \boxed{5} = \lim_{x\to 3} f(x)$

****

Definition: The **Derivative of a function** f **at a number** a, denoted by $f'(a)$, is given by

$$(*) \quad \boxed{f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}}$$

By the definition from class, this value is the slope of the tangent line **at** the given point $(a, f(a))$. This value captures the steepness of the curve **at** that point.

****

5. Suppose that $f(x) = 5 - 6x + 4x^2$.

(a) Compute $f'(1)$ using (*) above. (Here $a = 1$)

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{5 - 6(1+h) + 4(1+h)^2 - (5 - 6 + 4)}{h}$$
$$= \lim_{h \to 0} \frac{5 - 6 - 6h + 4 + 8h + 4h^2 - 5 + 6 - 4}{h} = \lim_{h \to 0} \frac{2h + 4h^2}{h}$$
$$= \lim_{h \to 0} \frac{h(2+4h)}{h} = \lim_{h \to 0} 2 + 4h = \boxed{2} \longleftarrow \text{ SLOPE!}$$

(b) Write the **equation of the tangent line** to the curve $y = f(x)$ at the point where $x = 1$.

From part (a) we know **slope**$=f'(1) = 2$.
(That is the slope of the tangent line to the curve $f(x)$ at the point where $x = 1$.)

The **point** is $(1, f(1)) = (1, 3)$. (You must compute the y value of the point.)

Using the point-slope form we have
$y - y_1 = m(x - x_1)$
$y - 3 = 2(x - 1)$
$\boxed{y = 2x + 1}$

****

If we replace a by a variable x above, we obtain the **derivative function** $f'(x)$ as

$$(**) \quad \boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

We will call this the *limit definition of the derivative*. Here $f'(x)$ is the function that takes in any value x and spits out the derivative at x. That is, the slope of the tangent line at the point $(x, f(x))$.

****

6. For each of the following, find $f'(x)$ using the *limit definition of the derivative* (**).

(a) $f(x) = x^3$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$
$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 = \boxed{3x^2}$$

(b) $f(x) = x^4$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h}$$
$$= \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h}$$
$$= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} = \lim_{h\to 0} \frac{h(4x^3 + 6x^2h + 4xh^2 + h^3)}{h}$$
$$= \lim_{h\to 0} 4x^3 + 6x^2h + 4xh^2 + h^3 = \boxed{4x^3}$$

(c) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$
$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$$

(d) $f(x) = \dfrac{1}{x}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{(x+h)(x)}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{x-x-h}{(x+h)(x)}}{h} = \lim_{h\to 0} \frac{-h}{(x+h)(x)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-1}{(x+h)(x)} = \boxed{-\frac{1}{x^2}}$$

(e) $f(x) = \dfrac{x+1}{x-1}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h}$$
$$= \lim_{h\to 0} \frac{\left(\frac{(x+h+1)(x-1) - (x+1)(x+h-1)}{(x+h-1)(x-1)} \right)}{h}$$
$$= \lim_{h\to 0} \frac{(x+h+1)(x-1) - (x+1)(x+h-1)}{(x+h-1)(x-1)} \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \frac{x^2 + xh + x - x - h - 1 - (x^2 + xh - x + x + h - 1)}{(x+h-1)(x-1)} \cdot \frac{1}{h}$$
$$= \lim_{h\to 0} \frac{x^2 + xh + x - x - h - 1 - x^2 - xh + x - x - h + 1}{(x+h-1)(x-1)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{-2h}{(x+h-1)(x-1)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)}$$

$$= \frac{-2}{(x-1)(x-1)} = \boxed{-\frac{2}{(x-1)^2}}$$

(f) $f(x) = \dfrac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}}\right)}{h} = \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x+h}\sqrt{x}} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} = \lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \cdot \left(\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}\right)$$

$$= \lim_{h\to 0} \frac{x-(x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0} \frac{x-x-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}$$

$$= \lim_{h\to 0} \frac{-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})} = \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x}+\sqrt{x+h})}$$

$$= \frac{-1}{\sqrt{x}\sqrt{x}(\sqrt{x}+\sqrt{x})} = \frac{-1}{x(2\sqrt{x})} = \boxed{-\frac{1}{2x^{\frac{3}{2}}}}$$