Math 105, Fall 2013

Worksheet 4, ANSWER KEY, Tuesday, October 1, 2013

- Please *show* all of your work and *justify* all of your answers.

1. Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim\limits_{x\to 7} \dfrac{x^2-4x-21}{x^2-3x} \overset{\text{DSP}}{=} \dfrac{0}{28} = \boxed{0}$

(b) $\lim\limits_{x\to 4} \dfrac{x^2-3x-4}{|x-4|} = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim\limits_{x\to 4^+} \dfrac{x^2-3x-4}{|x-4|} = \lim\limits_{x\to 4^+} \dfrac{x^2-3x-4}{x-4} = \lim\limits_{x\to 4^+} \dfrac{(x-4)(x+1)}{x-4} = \lim\limits_{x\to 4^+} x+1 = 5$

LHL: $\lim\limits_{x\to 4^-} \dfrac{x^2-3x-4}{|x-4|} = \lim\limits_{x\to 4^-} \dfrac{x^2-3x-4}{-(x-4)} = \lim\limits_{x\to 4^-} \dfrac{(x-4)(x+1)}{-(x-4)} = \lim\limits_{x\to 4^+} -(x+1) = -5$

Recall $|x-4| = \begin{cases} x-4 & \text{if } x-4 \geq 0 \\ -(x-4) & \text{if } x-4 < 0 \end{cases} = \begin{cases} x-4 & \text{if } x \geq 4 & \longleftarrow \text{RHL} \\ -(x-4) & \text{if } x < 4 & \longleftarrow \text{LHL} \end{cases}$

WARNING: The $|x-4|$ does not just cancel with the $x-4$. You must examine the two cases for the absolute value, because we are approaching 7.

(c) $\lim\limits_{x\to 4} \dfrac{x^2-2x-8}{x^2-5x+4}^{\left(\frac{0}{0}\right)} = \lim\limits_{x\to 4} \dfrac{(x-4)(x+2)}{(x-4)(x-1)} = \lim\limits_{x\to 4} \dfrac{x+2}{x-1} \overset{\text{DSP}}{=} \dfrac{6}{3} = \boxed{2}$

(d) $\lim\limits_{x\to -5} \dfrac{\frac{1}{1-x} - \frac{1}{6}}{x^2+3x-10}^{\left(\frac{0}{0}\right)} = \lim\limits_{x\to -5} \dfrac{\frac{6-(1-x)}{(1-x)6}}{x^2+3x-10} = \lim\limits_{x\to -5} \dfrac{5+x}{(1-x)6} \cdot \dfrac{1}{x^2+3x-10}$

$= \lim\limits_{x\to -5} \dfrac{5+x}{(1-x)6} \cdot \dfrac{1}{(x+5)(x-2)} = \lim\limits_{x\to -5} \dfrac{1}{(1-x)6(x-2)} \overset{\text{DSP}}{=} \dfrac{1}{(6)6(-7)} = \boxed{-\dfrac{1}{252}}$

(e) $\lim\limits_{x\to 3} \dfrac{x^2-12x+27}{x^2-6x+9}^{\left(\frac{0}{0}\right)} = \lim\limits_{x\to 3} \dfrac{(x-3)(x-9)}{(x-3)(x-3)} = \lim\limits_{x\to 3} \dfrac{x-9}{x-3}^{\left(\frac{-6}{0}\right)}$ $\boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim\limits_{x\to 3^+} \dfrac{x-9}{x-3} = \dfrac{-6}{0^+} = -\infty$

LHL: $\lim\limits_{x\to 3^-} \dfrac{x-9}{x-3} = \dfrac{-6}{0^-} = +\infty$

(f) $\lim_{x\to 3} \dfrac{x^2-12x+27}{x^2-9}^{\left(\frac{0}{0}\right)} = \lim_{x\to 3} \dfrac{(x-9)(x-3)}{(x-3)(x+3)} = \lim_{x\to 3} \dfrac{x-9}{x+3} \overset{\text{DSP}}{=} \dfrac{-6}{6} = \boxed{-1}$

(g) $\lim_{x\to 4} \dfrac{x+2}{4-x}^{\left(\frac{6}{0}\right)} = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 4^+} \dfrac{x+2}{4-x} = \dfrac{6}{0^-} = -\infty$

LHL: $\lim_{x\to 4^-} \dfrac{x+2}{4-x} = \dfrac{6}{0^+} = +\infty$

warning: watch the signs on the 0 piece in the denominator. We have $4-x$ here not $x-4$.

(h) $\lim_{x\to -4} \dfrac{x+2}{x+4} =^{\left(\frac{-2}{0}\right)}= \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to -4^+} \dfrac{x+2}{x+4} = \dfrac{-2}{0^+} = -\infty$

LHL: $\lim_{x\to -4^-} \dfrac{x+2}{x+4} = \dfrac{-2}{0^-} = +\infty$

(i) $\lim_{x\to 2} \dfrac{3-\sqrt{x+7}}{x^2-3x+2}^{\left(\frac{0}{0}\right)} = \lim_{x\to 2} \dfrac{3-\sqrt{x+7}}{x^2-3x+2} \cdot \dfrac{3+\sqrt{x+7}}{3+\sqrt{x+7}} = \lim_{x\to 2} \dfrac{9-(x+7)}{(x^2-3x+2)(3+\sqrt{x+7})}$

$= \lim_{x\to 2} \dfrac{2-x}{(x-2)(x-1)(3+\sqrt{x+7})} = \lim_{x\to 2} \dfrac{-(x-2)}{(x-2)(x-1)(3+\sqrt{x+7})}$

$= \lim_{x\to 2} \dfrac{-1}{(x-1)(3+\sqrt{x+7})} \overset{\text{L.L.}}{=} \dfrac{-1}{3+\sqrt{9}} = \boxed{-\dfrac{1}{6}}$

(j) $\lim_{x\to 1} \dfrac{G(x+2)+x-8}{G(2x)-3x^2-3x+2} =$ where $G(x) = (x-1)^2+3$

$\lim_{x\to 1} \dfrac{G(x+2)+x-8}{G(2x)-3x^2-3x+2} = \lim_{x\to 1} \dfrac{[((x+2)-1)^2+3]+x-8}{[(2x-1)^2+3]-3x^2-3x+2}$

$= \lim_{x\to 1} \dfrac{[(x+1)^2+3]+x-8}{[(2x-1)^2+3]-3x^2-3x+2} = \lim_{x\to 1} \dfrac{(x+1)^2+3)+x-8}{[(2x-1)^2+3]-3x^2-3x+2}$

$= \lim_{x\to 1} \dfrac{x^2+2x+1+3+x-8}{4x^2-4x+1+3-3x^2-3x+2} = \lim_{x\to 1} \dfrac{x^2+3x-4}{x^2-7x+6}^{\left(\frac{0}{0}\right)} = \lim_{x\to 1} \dfrac{(x+4)(x-1)}{(x-6)(x-1)}$

$= \lim_{x\to 1} \dfrac{x+4}{x-6} \overset{\text{DSP}}{=} \dfrac{5}{-5} = \boxed{-1}$

(k) $\lim_{x\to 7} \dfrac{x-7}{|7-x|} = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 7^+} \dfrac{x-7}{-(7-x)} = \lim_{x\to 7^+} \dfrac{x-7}{x-7} = 1$

LHL: $\lim_{x\to 7^-} \dfrac{x-7}{|7-x|} = \lim_{x\to 7^-} \dfrac{x-7}{7-x} = \lim_{x\to 7^-} \dfrac{x-7}{-(x-7)} = -1$

Recall $|7-x| = \begin{cases} 7-x & \text{if } 7-x \geq 0 \\ -(7-x) & \text{if } 7-x < 0 \end{cases} = \begin{cases} 7-x & \text{if } x \leq 7 & \longleftarrow \text{LHL} \\ -(7-x) & \text{if } x > 7 & \longleftarrow \text{RHL} \end{cases}$

WARNING: The $|7-x|$ does not just cancel with the $x-7$. You must examine the two cases for the absolute value. Plus, be careful with the signs. We have $7-x$ here and not $x-7$.

(l) $\lim_{x\to 5} \dfrac{f(x^2)-28}{(f(x))^2-10x-14} =$ where $f(x) = x+3$

$$\lim_{x\to 5} \frac{f(x^2)-28}{(f(x))^2-10x-14} = \lim_{x\to 5} \frac{x^2+3-28}{(x+3)^2-10x-14} = \lim_{x\to 5} \frac{x^2-25}{x^2+6x+9-10x-14}$$

$$= \lim_{x\to 5} \frac{x^2-25}{x^2-4x-5} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 5} \frac{(x-5)(x+5)}{(x-5)(x+1)} = \lim_{x\to 5} \frac{x+5}{x+1} \overset{\text{DSP}}{=} \frac{10}{6} = \boxed{\frac{5}{3}}$$

(m) $\lim_{x\to 2} \dfrac{x^2-9x+14}{x^2-4x+4} \overset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 2} \dfrac{(x-7)(x-2)}{(x-2)(x-2)} = \lim_{x\to 2} \dfrac{x-7}{x-2}$ $\boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x\to 2^+} \dfrac{x-7}{x-2} = \lim_{x\to 2^+} \dfrac{-5}{0^+} = -\infty$

LHL: $\lim_{x\to 2^-} \dfrac{x-7}{x-2} = \lim_{x\to 2^-} \dfrac{-5}{0^-} = +\infty$

(n) $\lim_{x\to 7} \dfrac{x^2-2x-35}{x^2-2x+1} \overset{\text{DSP}}{=} \dfrac{49-14-35}{49-14+1} = \dfrac{0}{36} = \boxed{0}$

2.

(a) Suppose that $f(x) = \sqrt{x}$. Compute the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

Simplify until you cancel the h in the denominator.

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = \boxed{\frac{1}{\sqrt{x+h}+\sqrt{x}}}$$

(b) Suppose that $f(x) = \sqrt{x^2-5x+3}$. Compute the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.

Simplify until you cancel the h in the denominator.

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{(x+h)^2-5(x+h)+3}-\sqrt{x^2-5x+3}}{h}$$

$$= \frac{\sqrt{(x+h)^2-5(x+h)+3}-\sqrt{x^2-5x+3}}{h} \cdot \left(\frac{\sqrt{(x+h)^2-5(x+h)+3}+\sqrt{x^2-5x+3}}{\sqrt{(x+h)^2-5(x+h)+3}+\sqrt{x^2-5x+3}}\right)$$

$$= \frac{[(x+h)^2-5(x+h)+3]-[x^2-5x+3]}{h(\sqrt{(x+h)^2-5(x+h)+3}+\sqrt{x^2-5x+3})} = \frac{x^2+2xh+h^2-5x-5h+3-x^2+5x-3}{h(\sqrt{(x+h)^2-5(x+h)+3}+\sqrt{x^2-5x+3})}$$

$$= \frac{2xh+h^2-5h}{h(\sqrt{(x+h)^2-5(x+h)+3}+\sqrt{x^2-5x+3})} = \frac{h(2x+h-5)}{h(\sqrt{(x+h)^2-5(x+h)+3}+\sqrt{x^2-5x+3})}$$

$$= \boxed{\frac{2x+h-5}{\sqrt{(x+h)^2-5(x+h)+3}+\sqrt{x^2-5x+3}}}$$

(c) Suppose that $f(x) = \dfrac{1-3x}{x+2}$. Compute the difference quotient $\dfrac{f(x+h)-f(x)}{h}$.
Simplify until you cancel the h in the denominator.

$$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{1-3(x+h)}{x+h+2}-\dfrac{1-3x}{x+2}}{h} = \frac{\dfrac{1-3x-3h}{x+h+2}-\dfrac{1-3x}{x+2}}{h}$$

$$= \frac{\dfrac{(1-3x-3h)(x+2)-(1-3x)(x+h+2)}{(x+h+2)(x+2)}}{h}$$

$$= \frac{\dfrac{x-3x^2-3xh+2-6x-6h-(x+h+2-3x^2-3xh-6x)}{(x+h+2)(x+2)}}{h}$$

$$= \frac{\dfrac{x-3x^2-3xh+2-6x-6h-x-h-2+3x^2+3xh+6x}{(x+h+2)(x+2)}}{h}$$

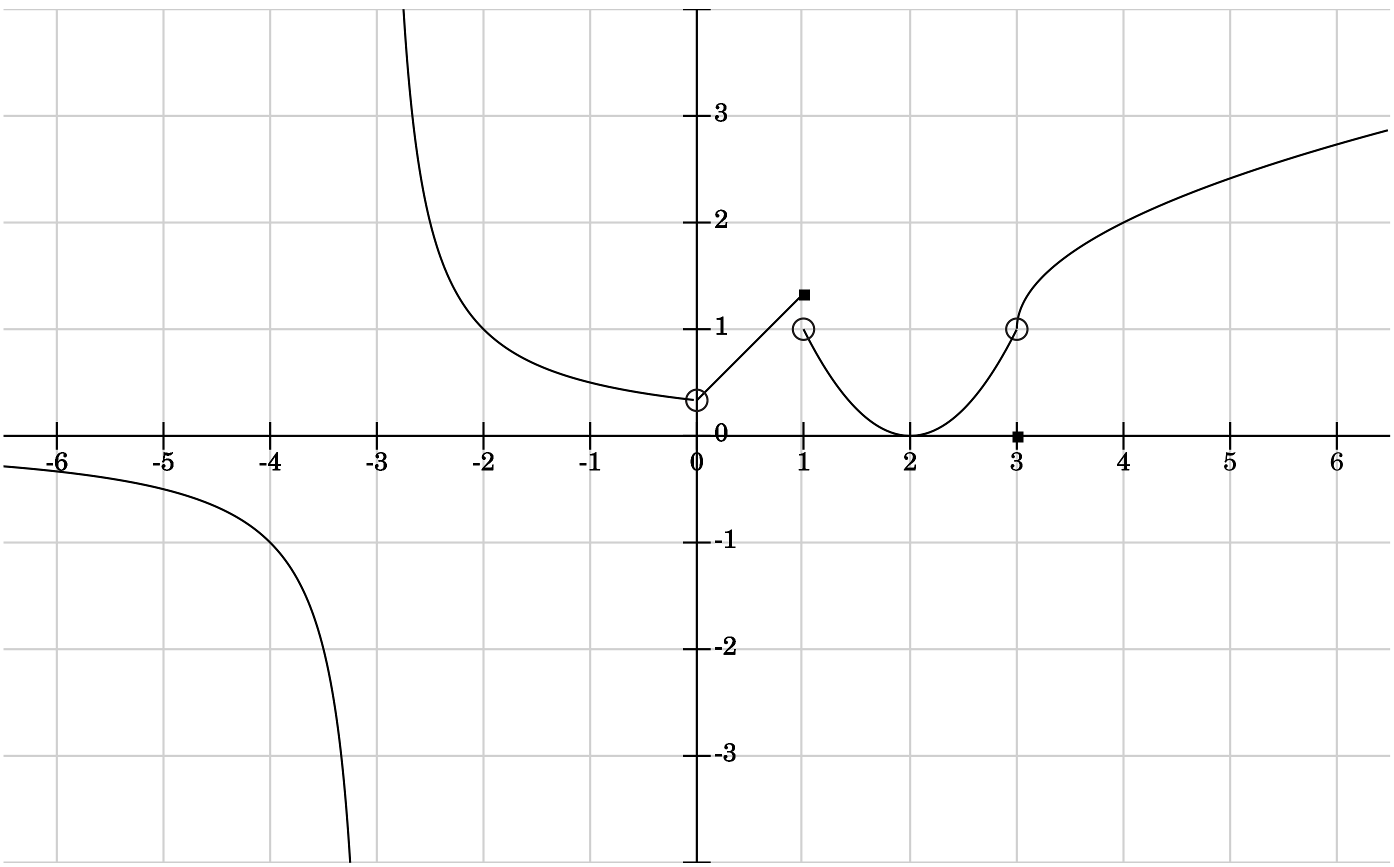
$$= \frac{\dfrac{-6h-h}{(x+h+2)(x+2)}}{h} = \frac{\dfrac{-7h}{(x+h+2)(x+2)}}{h} = \frac{-7h}{(x+h+2)(x+2)} \cdot \left(\frac{1}{h}\right)$$

$$= \boxed{\frac{-7}{(x+h+2)(x+2)}}$$

3. Consider the function defined by

$$f(x) = \begin{cases} \sqrt{x-3}+1 & \text{if } x > 3 \\ 0 & \text{if } x = 3 \\ (x-2)^2 & \text{if } 1 < x < 3 \\ x + \dfrac{1}{3} & \text{if } 0 < x \leq 1 \\ \dfrac{1}{x+3} & \text{if } x < 0 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$.



(b) State the **Domain** of the function $f(x)$. Domain= $\{x | x \neq 0, -3\}$

(c) Compute $\lim_{x \to -3} f(x) =$ DNE b/c RHL$\neq$ LHL

RHL: $\lim_{x \to -3^+} f(x) = +\infty$

LHL: $\lim_{x \to -3^-} f(x) = -\infty$

(d) Compute $\lim_{x \to 0} f(x) = \boxed{\dfrac{1}{3}}$

RHL: $\lim_{x \to 0^+} f(x) = \dfrac{1}{3}$

LHL: $\lim_{x \to 0^-} f(x) = \dfrac{1}{3}$

(e) Compute $\lim_{x \to 1} f(x) = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x \to 1^+} f(x) = 1$

LHL: $\lim_{x \to 1^-} f(x) = \dfrac{4}{3}$

(f) Compute $\lim_{x \to 3} f(x) = \boxed{1}$ RHL=LHL

RHL: $\lim_{x \to 3^+} f(x) = 1$

LHL: $\lim_{x \to 3^-} f(x) = 1$

(g) State all the value(s) at which f is discontinuous. Justify your answer(s) using the definition of continuity.

f is discontinuous at $x = -3, 0, 1, 3$. State the reasons WHY.

- $\boxed{f \text{ is discontinuous at } x = -3 \text{ because } \lim_{x \to -3} f(x) \text{ DNE}}$

- $\boxed{f \text{ is discontinuous at } x = 0 \text{ because } f(0) \text{ is undefined}}$

- $\boxed{f \text{ is discontinuous at } x = 1 \text{ because } \lim_{x \to 1} f(x) \text{ DNE}}$

- $\boxed{f \text{ is discontinuous at } x = 3 \text{ because } \lim_{x \to 3} f(x) \neq f(3)}$ **despite** the fact that $\lim_{x \to 3} f(x)$ exists AND $f(3) = 0$ is defined, they **are <u>NOT</u> equal**.

Recall: a function f is continuous at a number $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

That is, **three** things need to hold:

(1) $\lim_{x \to a} f(x)$ exists.

(2) $f(a)$ is defined.

(3) $\lim_{x \to a} f(x) = f(a)$ meaning (1) and (2) above are equal.