Math 105 Fall 2013

Worksheet 2, Tuesday, September 17, 2013

1. Let $g(x) = \dfrac{\dfrac{x}{x-1} - \dfrac{x+2}{x}}{x-2}$. Simplify $g(x)$. What is the domain of $g(x)$?

$$g(x) = \frac{\dfrac{x}{x-1} - \dfrac{x+2}{x}}{x-2} = \frac{\left(\dfrac{x^2-(x-1)(x+2)}{x(x-1)}\right)}{x-2}$$

$$= \frac{\left(\dfrac{x^2-(x^2+x-2)}{x(x-1)}\right)}{x-2} = \frac{\left(\dfrac{-x+2}{x(x-1)}\right)}{x-2} = \frac{\left(\dfrac{-x+2}{x(x-1)}\right)}{\dfrac{x-2}{1}}$$

$$= \left(\frac{-x+2}{x(x-1)}\right)\left(\frac{1}{x-2}\right) = \frac{-x+2}{x(x-1)(x-2)} = \frac{-(x-2)}{x(x-1)(x-2)} = \frac{-1}{x(x-1)}.$$

Domain $g = \{x : x \neq 0, 1, 2\}$.

2. Given two functions f and g. The **Composition** of f and g is defined by

$$f \circ g(x) = f(g(x))$$

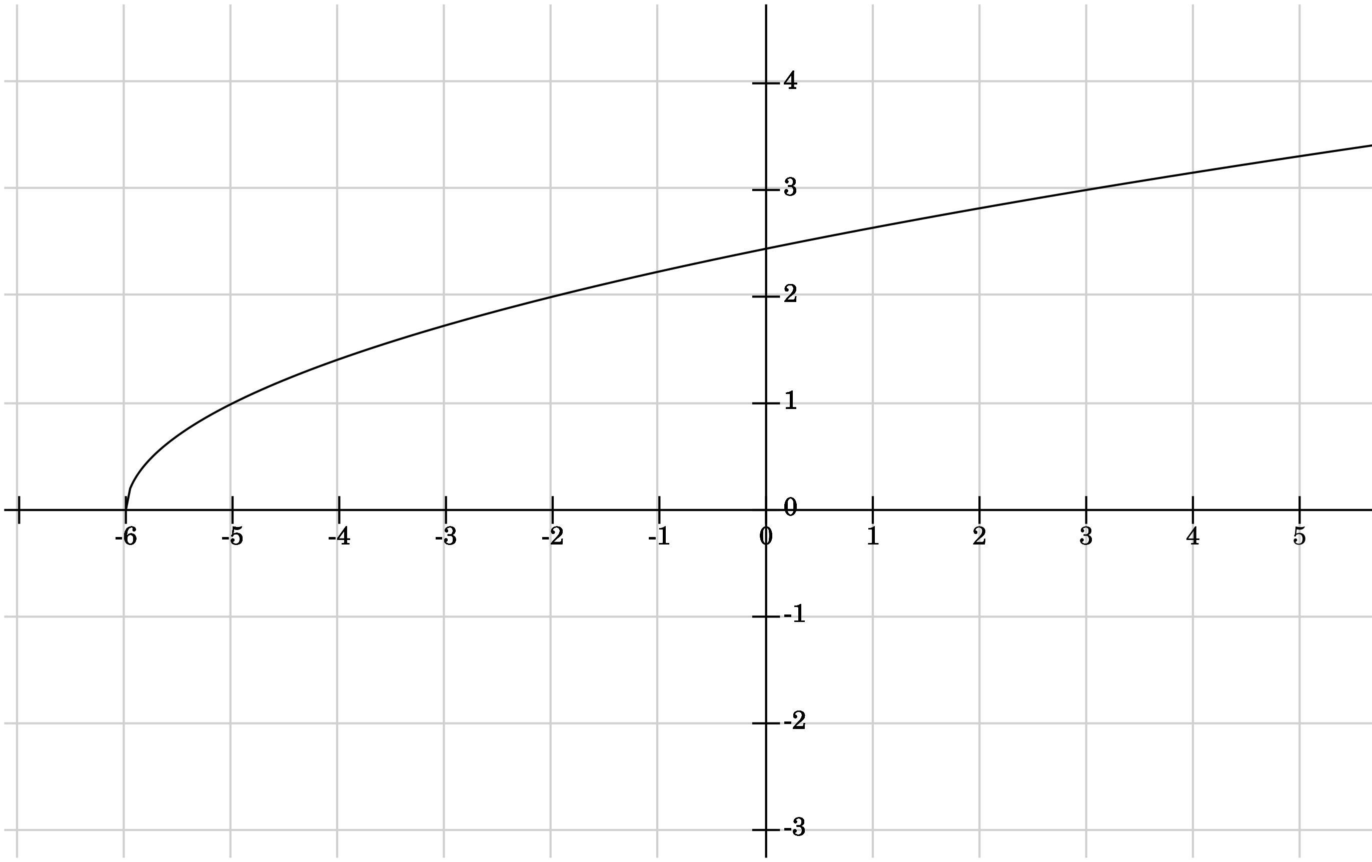
(a) Discuss what the Domain of $f \circ g$ is.

The Domain of the composite function is the set of all x values such that x is in the Domain of g and THEN that output $g(x)$ is in turn in the Domain of f.

(b) Take $f(x) = \sqrt{x+4}$ and $g(x) = x+2$. Compute **and** graph both $f \circ g$ and $g \circ f$. Discuss whether or not $f \circ g$ equals $g \circ f$. (Hint: what does it mean for two functions to be equal?)

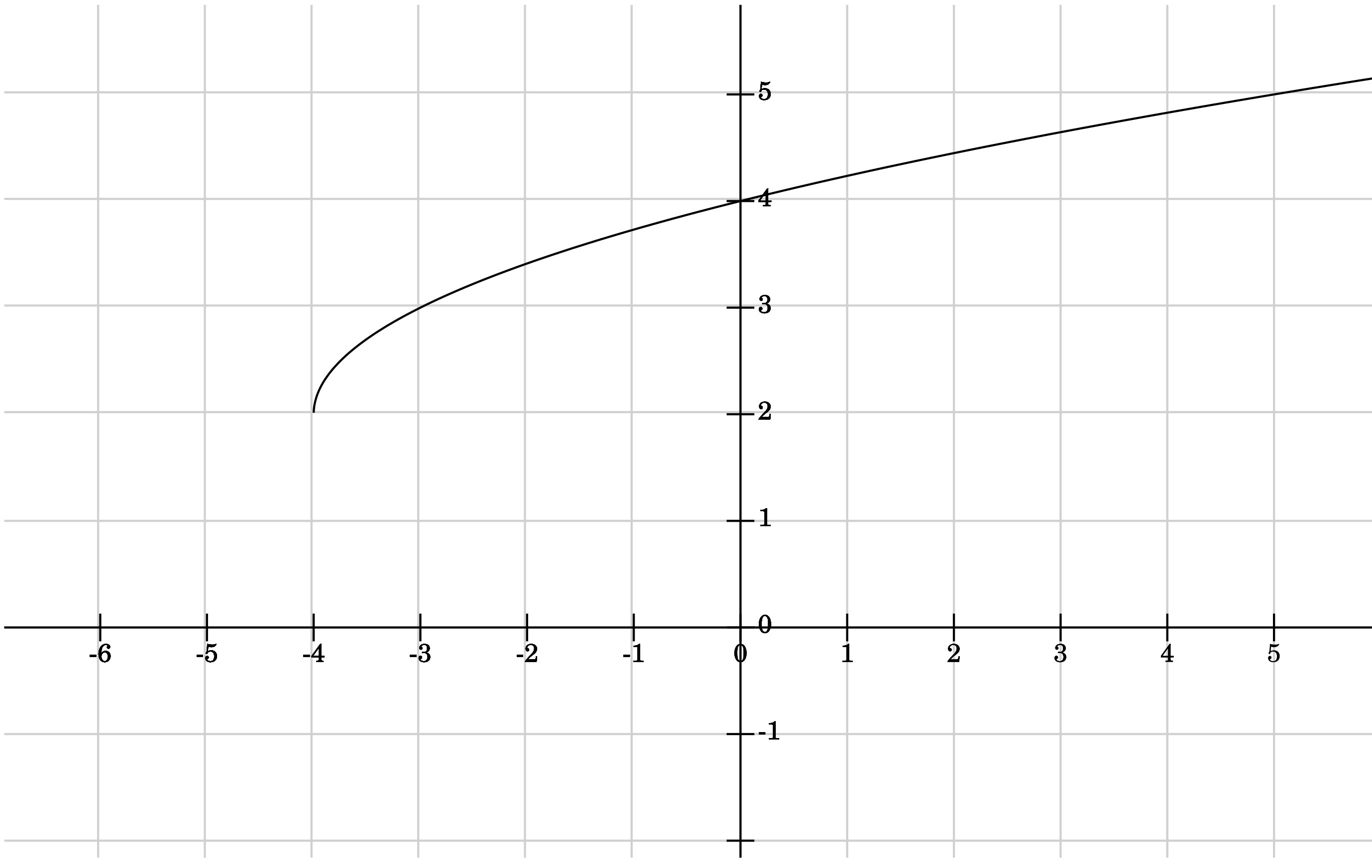
First, $f \circ g(x) = f(g(x)) = f(x+2) = \sqrt{(x+2)+4} = \boxed{\sqrt{x+6}}$.

This is the shift of the graph of $y = \sqrt{x}$ to the left 6 units.



Second, $g \circ f(x) = g(f(x)) = g(\sqrt{x+4}) = \boxed{\sqrt{x+4}+2}$.

This is the shift of the graph of $y = \sqrt{x}$ to the left 4 units and then up 2.



Notice that these are not the same functions. They don't have the same graphs. Equal functions must take the same value at every element of the domain. Notice that they also don't even have the same domains:

Domain $f \circ g = \{x : x \geq -6\}$ and Domain $g \circ f = \{x : x \geq -4\}$.

3. Let $f(x) = \dfrac{x+1}{x-1}$. Compute $f(f(2))$. Compute and simplify $f(f(x))$. Hint: first find a large formula for $f(f(x))$. Then simplify by finding common denominators.

First $f(2) = \dfrac{2+1}{2-1} = \dfrac{3}{1} = 3$. Then $f(f(2)) = f(3) = \dfrac{3+1}{3-1} = \dfrac{4}{2} = \boxed{2}$.

Next, $f(f(x)) = f\left(\dfrac{x+1}{x-1}\right) = \dfrac{\dfrac{x+1}{x-1}+1}{\dfrac{x+1}{x-1}-1} = \dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x-1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x-1}}$

$$= \frac{\dfrac{x+1+x-1}{x-1}}{\dfrac{x+1-(x-1)}{x-1}} = \frac{\dfrac{x+1+x-1}{x-1}}{\dfrac{x+1-x+1}{x-1}} = \frac{\dfrac{2x}{x-1}}{\dfrac{2}{x-1}} = \left(\frac{2x}{x-1}\right)\left(\frac{x-1}{2}\right) = \boxed{x}$$

4. Let $f(x) = \dfrac{1}{x+1}$. Compute and simplify $\dfrac{f(x+h)-f(x)}{h}$.

warning: $f(x+h) \neq f(x) + h$ **be careful!**

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} = \frac{\frac{(x+1)}{(x+1)(x+h+1)} - \frac{(x+h+1)}{(x+h+1)(x+1)}}{h}$$

$$= \frac{\frac{(x+1)-(x+h+1)}{(x+1)(x+h+1)}}{h} = \frac{\frac{x+1-x-h-1}{(x+1)(x+h+1)}}{h}$$

$$= \frac{\frac{-h}{(x+1)(x+h+1)}}{h} = \frac{-h}{h(x+1)(x+h+1)} = \boxed{\frac{-1}{(x+1)(x+h+1)}}$$

Tip: Leave the denominator factored, in case something cancels.

5. Let $f(x) = \frac{x-7}{x+3}$. Compute and simplify $\frac{f(x+h)-f(x)}{h}$.

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h-7}{x+h+3} - \frac{x-7}{x+3}}{h} = \frac{\frac{(x+h-7)(x+3)}{(x+3)(x+h+3)} - \frac{(x-7)(x+h+3)}{(x+3)(x+h+3)}}{h}$$

$$= \frac{\frac{(x+h-7)(x+3)-(x-7)(x+h+3)}{(x+3)(x+h+3)}}{h}$$

$$= \frac{\frac{x^2+xh-7x+3x+3h-21-(x^2+xh+3x-7x-7h-21)}{(x+3)(x+h+3)}}{h}$$

$$= \frac{\frac{x^2+xh-7x+3x+3h-21-x^2-xh-3x+7x+7h+21}{(x+3)(x+h+3)}}{h}$$

$$= \frac{\frac{3h+7h}{(x+3)(x+h+3)}}{h} = \frac{\frac{10h}{(x+3)(x+h+3)}}{h} = \frac{\frac{10h}{(x+3)(x+h+3)}}{\frac{h}{1}}$$

$$= \frac{10h}{(x+3)(x+h+3)}\left(\frac{1}{h}\right)$$

$$= \boxed{\frac{10}{(x+3)(x+h+3)}}$$

Notice all non-h terms should cancel. Watch your algebra.

6. Simplify each of the following expressions.

(a) $\frac{x^2+6x+8}{x^2-4} = \frac{(x+2)(x+4)}{(x+2)(x-2)} = \boxed{\frac{x+4}{x-2}}$

(b) $\frac{x^2+6x+8}{x^2-5x-14} = \frac{(x+2)(x+4)}{(x+2)(x-7)} = \boxed{\frac{x+4}{x-7}}$

(c) $\dfrac{x^2-6x+8}{x^2-x-2} = \dfrac{(x-2)(x-4)}{(x-2)(x+1)} = \boxed{\dfrac{x-4}{x+1}}$

(d) $\dfrac{1}{t\sqrt{1+t}} - \dfrac{1}{t} = \dfrac{1}{t\sqrt{1+t}} - \dfrac{\sqrt{1+t}}{t\sqrt{1+t}} = \boxed{\dfrac{1-\sqrt{1+t}}{t\sqrt{1+t}}}$

(e) $\dfrac{t-1}{g(t^2)-3}$, where $g(t) = 2t+1$

$$\frac{t-1}{g(t^2)-3} = \frac{t-1}{(2t^2+1)-3} = \frac{t-1}{2t^2-2} = \frac{t-1}{2(t^2-1)} = \frac{t-1}{2(t-1)(t+1)} = \boxed{\frac{1}{2(t+1)}},$$

(f) $\dfrac{x^2-13x+42}{x^2-4x+12} = \dfrac{(x-7)(x-6)}{(x+2)(x-6)} = \boxed{\dfrac{x-7}{x+2}}$

(g) $\dfrac{1}{x} - \dfrac{1}{|x|}$ Hint: you might need two cases here. Write out the definition of $|x|$.

Recall $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Case $x > 0$: $\dfrac{1}{x} - \dfrac{1}{|x|} = \dfrac{1}{x} - \dfrac{1}{x} = \boxed{0}$

Case $x < 0$: $\dfrac{1}{x} - \dfrac{1}{|x|} = \dfrac{1}{x} - \dfrac{1}{-x} = \dfrac{1}{x} + \dfrac{1}{x} = \boxed{\dfrac{2}{x}}$

(h) $\dfrac{|x+4|}{x+4}$ Hint: you might need two cases here. Write out the definition of $|x+4|$.

Recall $|x+4| = \begin{cases} x+4 & \text{if } x+4 \geq 0 \\ -(x+4) & \text{if } x+4 < 0 \end{cases} = \begin{cases} x+4 & \text{if } x \geq -4 \\ -(x+4) & \text{if } x < -4 \end{cases}$

Case $x > -4$: $\dfrac{|x+4|}{x+4} = \dfrac{x+4}{x+4} = \boxed{1}$

Case $x < -4$: $\dfrac{|x+4|}{x+4} = \dfrac{-(x+4)}{x+4} = \boxed{-1}$

(i) Let $f(x) = \dfrac{1}{x}$. Compute and simplify $\dfrac{f(t-1)-2f(t)}{t^2-4}$

$$\frac{f(t-1)-2f(t)}{t^2-4} = \frac{\frac{1}{t-1} - \frac{2}{t}}{t^2-4} = \frac{\frac{1}{t-1}\left(\frac{t}{t}\right) - \frac{2}{t}\left(\frac{t-1}{t-1}\right)}{t^2-4} = \frac{\left(\frac{t-2(t-1)}{(t-1)t}\right)}{t^2-4}$$

$$= \frac{\left(\frac{t-2t+2)}{(t-1)t}\right)}{t^2-4} = \frac{\left(\frac{-t+2}{(t-1)t}\right)}{t^2-4}$$

$$= \frac{\left(\frac{-t+2}{(t-1)t}\right)}{\frac{t^2-4}{1}} = \left(\frac{-t+2}{(t-1)t}\right)\left(\frac{1}{t^2-4}\right) = \left(\frac{-(t-2)}{(t-1)t}\right)\left(\frac{1}{(t-2)(t+2)}\right)$$

$$= \boxed{\left(\frac{-1}{(t-1)t(t+2)} \right)}$$

Recall that $\lim\limits_{x \to a^+} f(x) = L$ is the **Right Hand Limit**. We will denote this as **RHL**. It means you are studying the limiting value (if it exists) of the output function values as the input values approach a *from the right of* a. Also recall that $\lim\limits_{x \to a^-} f(x) = L$ is the **Left Hand Limit**. We will denote this as **LHL**. It means you are studying the limiting value (if it exists) of the output function values as the input values approach a *from the left of* a.

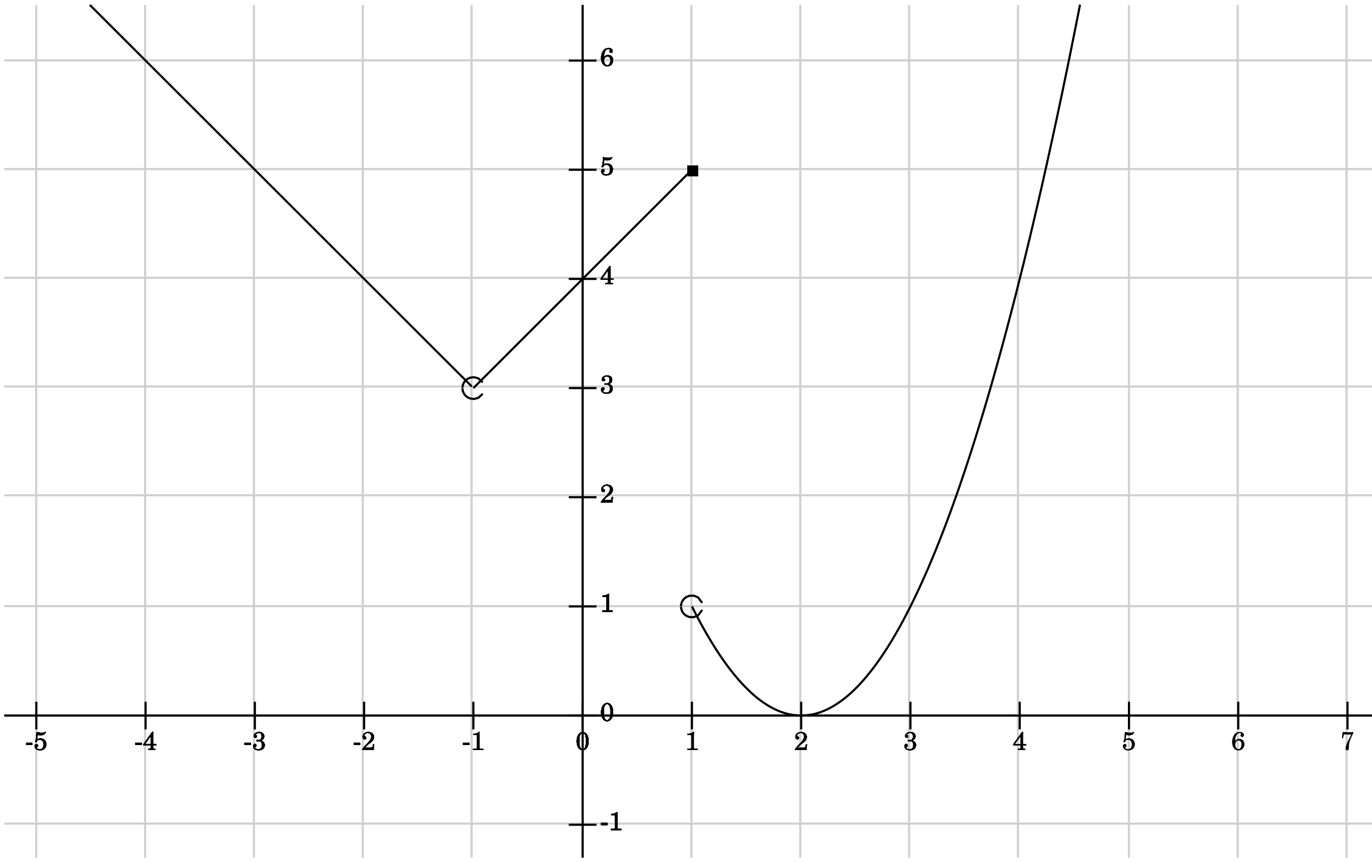
We write $\lim\limits_{x \to a} f(x) = L$ to represent the full **two-sided limit**. We have the following result.

Theorem: $\boxed{\lim\limits_{x \to a} f(x) = L \text{ if and only if } \lim\limits_{x \to a^+} f(x) = L \text{ and } \lim\limits_{x \to a^-} f(x) = L.}$

That is, the two-sided limit exists at a if and only if **both** the one-sided limits, from the right and from the left, exist and are equal. If RHL$\neq$ LHL then we say the two-sided limit Does Not Exist or **DNE**.

7. Consider the function defined piece-wise by $f(x) = \begin{cases} 2-x & \text{if } x < -1 \\ x+4 & \text{if } -1 < x \leq 1 \\ (x-2)^2 & \text{if } x > 1 \end{cases}$

   Graph $f(x)$ carefully and find its Domain and Range. Compute $\lim\limits_{x \to 1^+} f(x)$. Compute $\lim\limits_{x \to 1^-} f(x)$. Determine whether $\lim\limits_{x \to 1} f(x)$ exists.



It appears that $f(x)$ is approaching 1 *from the right* of 1 and approaching 5 *from the left* of 1. That is,

RHL: $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-2)^2 = \boxed{1}$

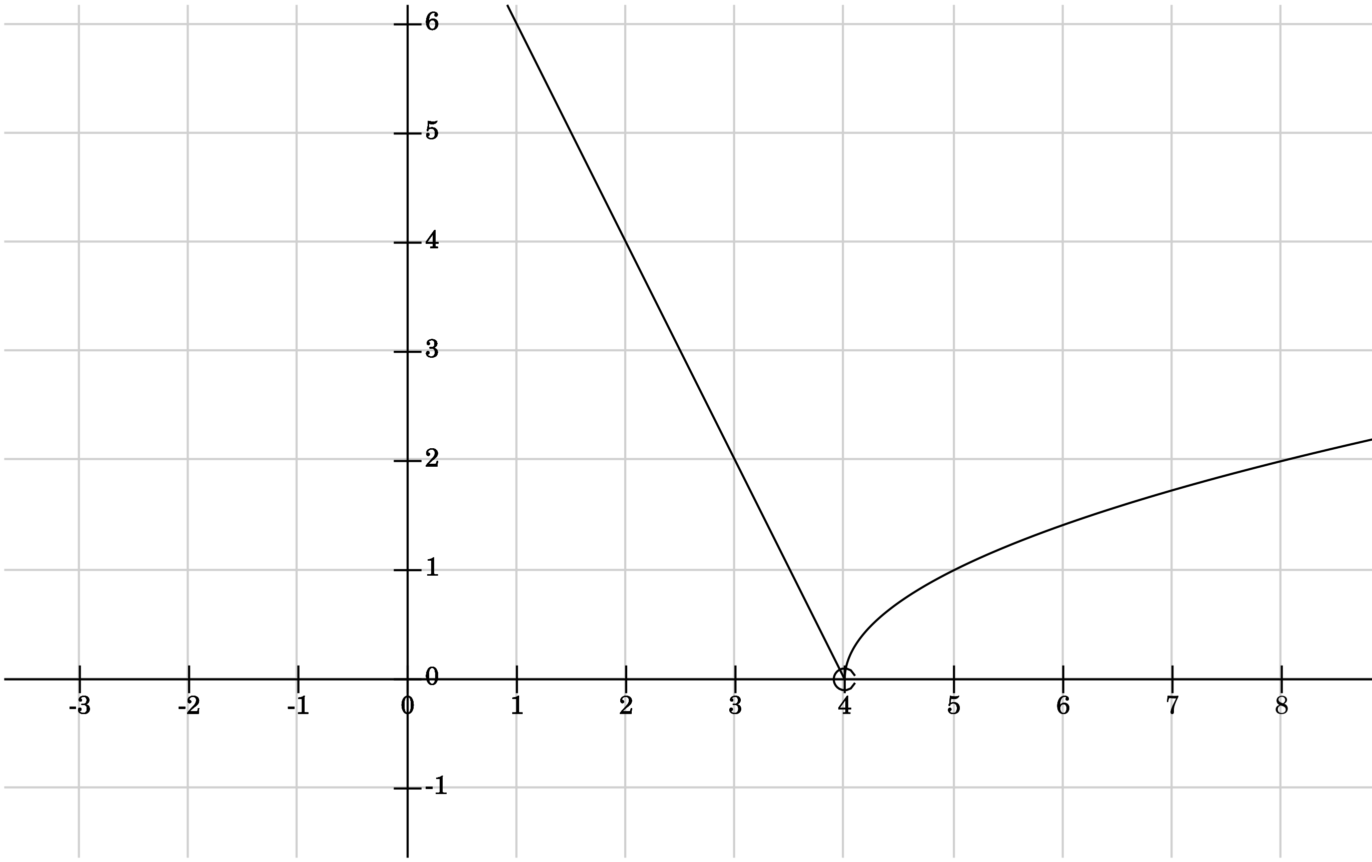
LHL: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x+4 = \boxed{5}$

Finally, we conclude that $\lim_{x\to 1} f(x)$ $\boxed{\text{Does Not Exist}}$ since RHL$\neq$LHL.

8. Consider the function defined by

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } \ x > 4 \\ 8-2x & \text{if } \ x < 4 \end{cases}$$

Determine whether $\lim_{x\to 4} f(x)$ exists? Why or why not? Hint: You need to investigate one-sided limits on your own...



It appears that $f(x)$ is approaching 0 *from the right* of 4 and approaching 0 *from the left* of 4. That is,

RHL: $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \sqrt{x-4} = \boxed{0}$

LHL: $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 8-2x = \boxed{0}$

Finally, we conclude that $\lim_{x\to 4} f(x) = 0$ since RHL=LHL=0.