Name:____________________________

Math 105 | Quiz #9 | December 2, 2013

Answer Key

1. [10 Points] Let $f(x) = x^4 + 4x^3$. For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

• Domain: $f(x)$ has domain $(-\infty, \infty)$

• VA: It is a polynomial, continuous everywhere, and so has no vertical asymptotes.

• HA: There are no horizontal asymptotes for this f since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$ because

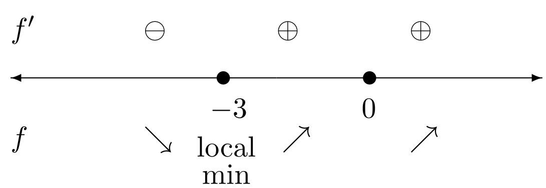
$\lim_{x\to\infty} x^4+4x^3 = \lim_{x\to\infty} x^3(x+4) = \infty\cdot\infty = \infty$ and $\lim_{x\to-\infty} x^4+4x^3 = \lim_{x\to-\infty} x^3(x+4) = (-\infty)\cdot(-\infty) = \infty$.

• First Derivative Information:

We compute $f'(x) = 4x^3 + 12x^2$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when

$$4x^3 + 12x^2 = 4x^2(x+3) = 0 \Longrightarrow x = 0 \text{ or } x = -3$$

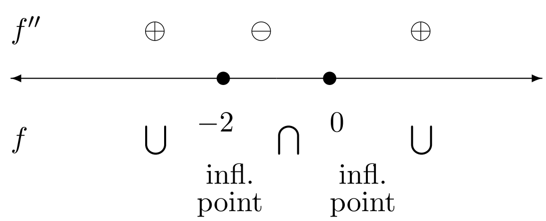
As a result, $x = 0$ and $x = -3$ are the critical numbers. Using sign testing/analysis for f',



So f is increasing on $(-3, \infty)$; and f is decreasing on $(-\infty, -3)$. Moreover, f has a local min at $x = -3$ with $f(-3) = -27$.

• Second Derivative Information:

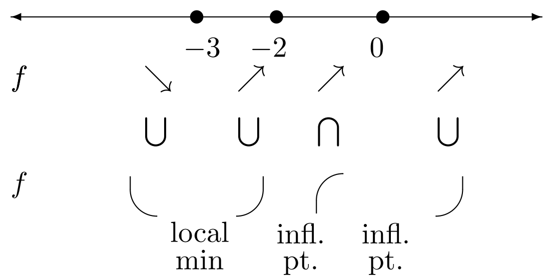
Meanwhile, f'' is always defined and continuous, and $f'' = 12x^2 + 24x = 12x(x+2) = 0$ only at our possible inflection points $x = 0$ and $x = -2$. Using sign testing/analysis for f'',



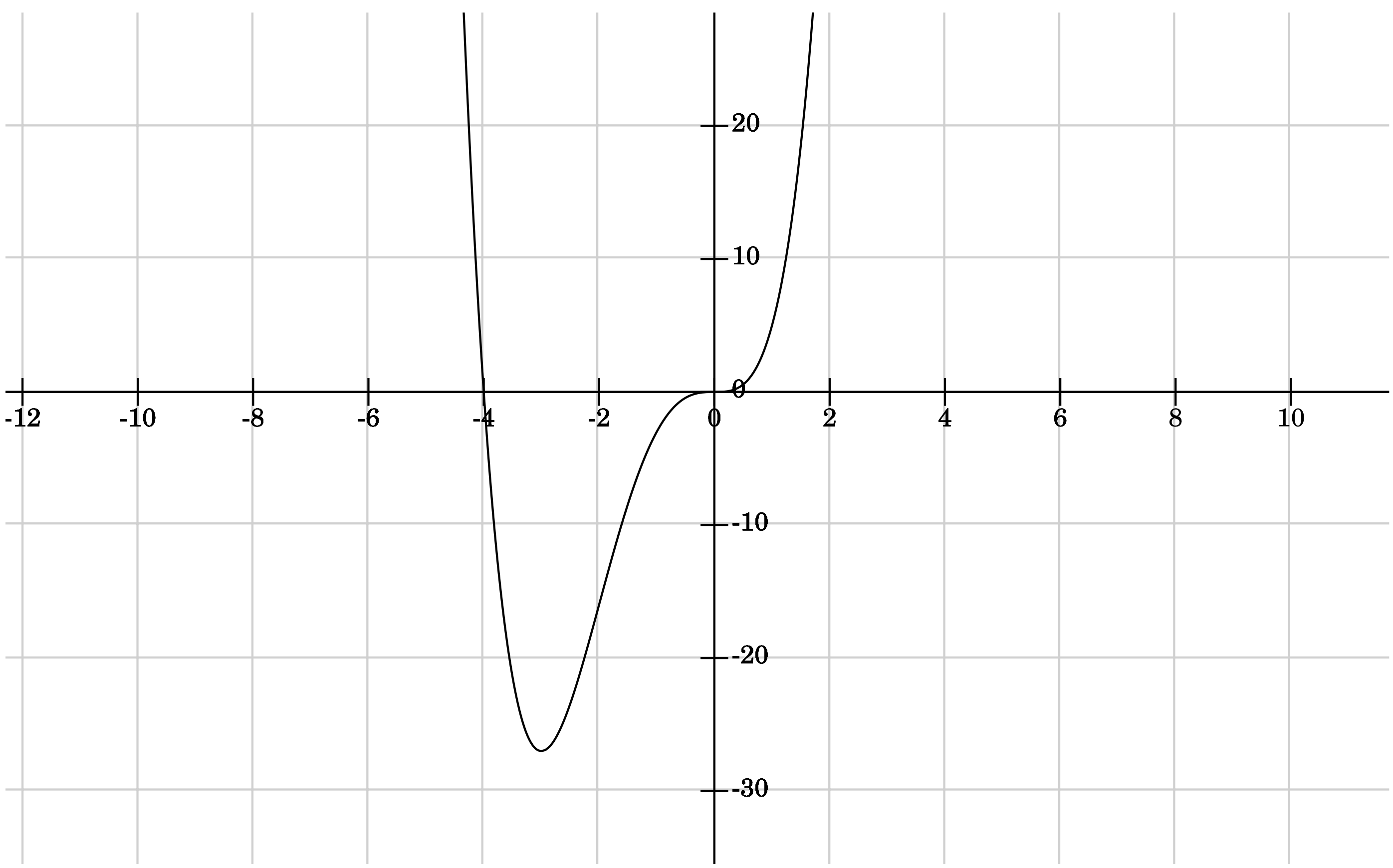
So f is concave down on $(-2, 0)$ and concave up on $(-\infty, -2)$ and $(0, \infty)$, with inflection points at

$x = 0$ and $x = -2$ with $f(0) = 0$ and $f(-2) = -16$.

- Piece the first and second derivative information together:



- Sketch:



2. [10 Points] Let $f(x) = \dfrac{x^2-1}{x^2-4}$. For this function, discuss domain, vertical and horizontal asymptote(s), interval(s) of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve. **Hint:**

Take my word for it that (you do **not** have to compute these)

$$\boldsymbol{f'(x) = \frac{-6x}{(x^2-4)^2}} \quad \text{and} \quad \boldsymbol{f''(x) = \frac{6(3x^2+4)}{(x^2-4)^3}}.$$

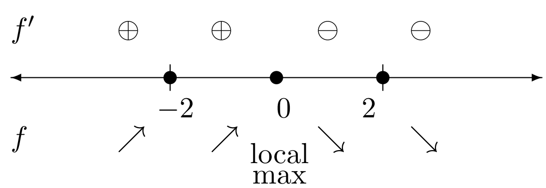
- Domain: $f(x)$ has domain $\{x | x \neq \pm 2\}$
- VA: Vertical asymptotes at $x = \pm 2$.
- HA: Horizontal asymptote at $y = 1$ for this f since

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2-1}{x^2-4} = \lim_{x\to\pm\infty} \frac{x^2-1}{x^2-4} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\pm\infty} \frac{1-\frac{1}{x^2}}{1-\frac{4}{x^2}} = \frac{1}{1} = 1.$$

- First Derivative Information:

We were given $f'(x) = \dfrac{-6x}{(x^2-4)^2}$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is zero or undefined. The former happens when $x = 0$. The derivative is undefined when $x = \pm 2$, but those values are not in the domain of the original function. As a result, $x = 0$ is technically the only critical number.

Using sign testing/analysis for f',

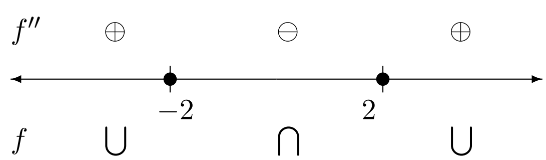


So f is decreasing on $(0, 2)$ and $(2, \infty)$; and f is increasing on $(-\infty, -2)$ and $(-2, 0)$. Moreover, f has a local max at $x = 0$ with $f(0) = \dfrac{1}{4}$.

- Second Derivative Information:

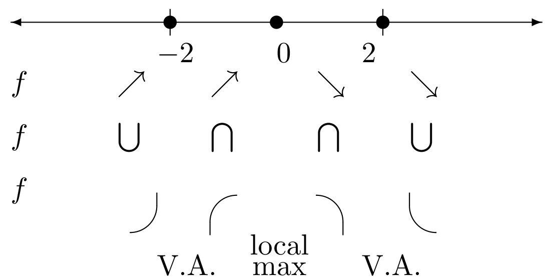
We were given $f''(x) = \dfrac{6(3x^2+4)}{(x^2-4)^2}$ which is never zero since $3x^2 + 4 \neq 0$.

Using sign testing/analysis for f'' around the vertical asymptotes,



So f is concave up on $(-\infty, -2)$ and $(2, \infty)$ and concave down on $(-2, 2)$ with no technical inflection points, since $x = \pm 2$ not in domain of original function.

- Piece the first and second derivative information together:



- Sketch: