HOMEWORK #23

Review Packet for Exam #3

Due Wednesday December 4 at the beginning of class.

Answer Key

Critical Numbers

1. Find critical numbers for the function $f(x) = x\sqrt{6-x}$.

   First compute the derivative

   $f'(x) = x\dfrac{1}{2\sqrt{6-x}}(-1) + \sqrt{6-x}\ (1) = x\dfrac{1}{2\sqrt{6-x}}(-1) + \sqrt{6-x}\left(\dfrac{2\sqrt{6-x}}{2\sqrt{6-x}}\right)$

   $\dfrac{-x}{2\sqrt{6-x}} + \dfrac{2(6-x)}{2\sqrt{6-x}} = \dfrac{-x+12-2x}{2\sqrt{6-x}} = \dfrac{-3x+12}{2\sqrt{6-x}}.$

   Critical numbers are where the derivative equals 0 or is undefined.

   First $f'(x) = \dfrac{-3x+12}{2\sqrt{6-x}} = 0$ when the numerator $-3x+12$ equals 0, which is when $x = 4$

   Second the derivative is undefined here where the denominator is 0, which is when $x = 6$, which we should note **is** in the domain of the original function.

   Finally the critical numbers are $\boxed{x = 4}$ and $\boxed{x = 6}$.

2. Find critical numbers for the function $f(x) = x^{\frac{5}{4}} - 5x^{\frac{1}{4}}$.

   First $f'(x) = \dfrac{5}{4}x^{\frac{1}{4}} - \dfrac{5}{4}x^{-\frac{3}{4}} = \dfrac{5}{4}x^{\frac{1}{4}} - \dfrac{5}{4x^{\frac{3}{4}}} = \left(\dfrac{x^{\frac{3}{4}}}{x^{\frac{3}{4}}}\right)\dfrac{5}{4}x^{\frac{1}{4}} - \dfrac{5}{4x^{\frac{3}{4}}}$

   $= \dfrac{5x}{4x^{\frac{3}{4}}} - \dfrac{5}{4x^{\frac{3}{4}}} = \dfrac{5x-5}{4x^{\frac{3}{4}}} = 0$

   when the numerator equals 0, which is when $5x - 5 = 0$ or when $x = 1$.

   Secondly the derivative is underfined when the denominator equals 0 here, when $x = 0$, which was in the domain of the original function.

   Finally the critical numbers are $\boxed{x = 1 \text{ and } x = 0.}$

Absolute Extreme Values

3. Find the absolute maximum and absolute minimum values of

   $$h(x) = \frac{x^2-1}{x^2+1} \text{ on } [-1, 3].$$

   $h'(x) = \dfrac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2} = \dfrac{(x^2+1)(2x) - (x^2-1)(2x)}{(x^2+1)^2}$

   $= \dfrac{2x^3+2x-2x^3-2x}{(x^2+1)^2} = \dfrac{4x}{(x^2+1)^2}.$

On the interval $[-1, 3]$, h' is always defined (since it is a rational function and its denominator is never zero). Also, $h'(x) = 0$ happens only when the numerator is zero; that is, when $4x = 0$, so $x = 0$. Applying the closed interval method:

$h(0) = -1 \longleftarrow$Absolute Minimum Value

$h(-1) = 0$

$h(3) = \frac{4}{5} \longleftarrow$Absolute Maximum Value

So the absolute maximum value is $\frac{4}{5}$ (attained at $x = 3$), and the absolute minimum value is -1 (attained at $x = 0$).

4. Find the absolute maximum and absolute minimum values of

$$f(x) = (x-4)^2(x+2)^2 \text{ on } [0, 5].$$

$f'(x) = (x-4)^2 \cdot 2(x+2) + (x+2)^2 \cdot 2(x-4) = 2(x-4)(x+2)[x-4+x+2] = 2(x-4)(x+2)[2x-2]$. On the interval $[0, 5]$, f' is always defined. Also, $f'(x) = 0$ happens only when $x = 4$, $x = -2$, and $x = 1$ (our critical numbers). Here $x = -2$ is outside of our interval of interest. Applying the closed interval method:

$f(1) = \boxed{81} \longleftarrow$ Absolute Maximum Value

$f(4) = \boxed{0} \longleftarrow$ Absolute Minimum Value

$f(0) = 64$

$f(5) = 49$.

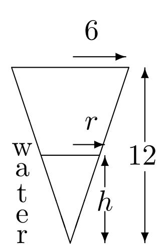
So the absolute maximum value is 81 (attained at $x = 1$), and the absolute minimum value is 0 (attained at $x = 4$).

Related Rates

5. A conical reservoir, 12 ft. deep and also 12 ft. across the top is being filled with water at the rate of 5 cubic feet per minute. How fast is the water rising when it is 4 feet deep?

The cross section (with water level drawn in) looks like:

• Diagram



• Variables
Let r = radius of the water level at time t
Let h = height of the water level at time t
Let V = volume of the water in the tank at time t
Find $\frac{dh}{dt} = ?$ when $h = 4$ feet
and $\frac{dV}{dt} = 5 \frac{\text{ft}^3}{\text{sec}}$

• Equation relating the variables:

Volume= $V = \frac{1}{3}\pi r^2 h$

• Extra solvable information: Note that r is not mentioned in the problem's info. But there is a relationship, via similar triangles, between r and h. We must have

$$\frac{r}{6} = \frac{h}{12} \implies r = \frac{h}{2}$$

After substituting into our previous equation, we get:

$V = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{1}{12}\pi h^3\right) \implies \frac{dV}{dt} = \frac{1}{12}\pi \cdot 3h^2 \cdot \frac{dh}{dt} \implies \frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$5 = \frac{1}{4}\pi(4)^2 \frac{dh}{dt}$

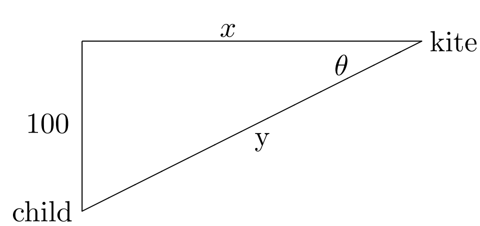
• Solve for the desired quantity:

$\frac{dh}{dt} = \frac{5 \cdot 4}{16\pi} = \frac{5}{4\pi} \frac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The water is rising at a rate of $\frac{5}{4\pi}$ feet every second at that moment.

6. A kite 100 feet high is being blown horizontally at 8 feet per second. When there are 300 feet of string out, how fast is the string running out?

• Diagram



The picture at arbitrary time t is:

• Variables

Let x = distance kite has travelled horizontally at time t

Let y = distance between kite and child at time t

Find $\frac{dy}{dt}$ =? when $y = 300$ feet

and $\frac{dx}{dt} = 8 \frac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

We have $x^2 + 100^2 = y^2$ by the Pythagorean Theorem.

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(x^2+100^2) = \frac{d}{dt}(y^2) \implies 2x\frac{dx}{dt} = 2y\frac{dy}{dt} \implies x\frac{dx}{dt} = y\frac{dy}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

At the key instant when $y = 300$, using the original equation, we have $x = \sqrt{(300)^2 - (100)^2}$ $= \sqrt{80000} = 200\sqrt{2}$.

So, $x\frac{dx}{dt} = y\frac{dy}{dt}$

and then $200\sqrt{2} \cdot 8 = 300\frac{dy}{dt}$

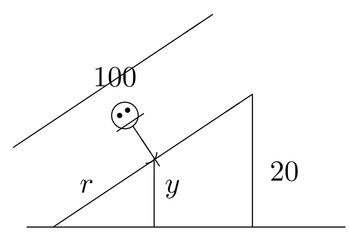
• Solve for the desired quantity:

$\frac{dy}{dt} = \frac{1600\sqrt{2}}{300} = \frac{80\sqrt{2}}{15}\frac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The string is running out at a rate of $\frac{16\sqrt{2}}{3}$ feet every second at that moment.

7. A waterskier skis up over the ramp at a speed of 30 ft./sec. The 100 ft. ramp slopes straight from no height at one end to 20 feet on the other end. How fast is she rising vertically just as she leaves the ramp?

• Diagram



• Variables

Let r = distance skiier up the ramp at time t

Let y = height of the skiier (up the ramp) above water level at time t

Find $\frac{dy}{dt}$ =? when $y = 20$

and $\frac{dr}{dt} = 30\frac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

Using similar triangles: $\frac{y}{20} = \frac{r}{100}$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}\left(\frac{y}{20}\right) = \frac{d}{dt}\left(\frac{r}{100}\right) \implies \frac{1}{20}\frac{dy}{dt} = \frac{1}{100}\frac{dr}{dt} \implies \frac{dy}{dt} = \frac{1}{5}\frac{dr}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$\frac{dy}{dt} = \frac{1}{5}(30)$

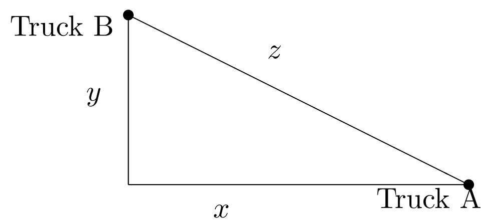
• Solve for the desired quantity:

$\frac{dy}{dt} = 6\frac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The skiier is rising vertically at a rate of 6 feet every second at that moment. (Note that this was independent of r.)

8. Two trucks leave a depot at the same time. Truck A travels east at 40 miles per hour, while Truck B travels north at 30 miles per hour. How fast is the distance between the trucks changing 60 minutes after leaving the depot?

• Diagram



• Variables
Let x = distance Truck A travelled East at time t
Let y = distance Truck B travelled North at time t
Let z = distance between Trucks A and B at time t
Find $\frac{dz}{dt}$ =? after 1 hour, when $x = 40$ miles, $y = 30$, $\frac{dx}{dt} = 40$ m.p.h.
and $\frac{dy}{dt} = 30$ m.p.h.

• Equation relating the variables:

Pythagorean Theorem gives $x^2 + y^2 = z^2$

• Differentiate both sides w.r.t. time t.

$$\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(z^2) \implies 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt} \implies x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt} \text{ (Related Rates!)}$$

• Substitute Key Moment Information (now and not before now!!!):

First, note that by the Pythagorean Theorem $z = \sqrt{(40)^2 + (30)^2} = 50$, so

$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$ becomes

$$40(40) + 30(30) = 50\frac{dz}{dt}$$

• Solve for the desired quantity:

$$\frac{dz}{dt} = \frac{1600 + 900}{50} = 50 \text{ m.p.h.}$$

• Answer the question that was asked: The distance between the trucks is increasing at a rate of 50 miles every hour at that moment.

Limits Evaluate the following limits. Please show your work.

9. $$\lim_{x\to-\infty} \frac{x^3 - 2x}{4x^3 + 1} = \lim_{x\to-\infty} \frac{x^3 - 2x}{4x^3+1} \cdot \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)} = \lim_{x\to-\infty} \frac{1 - \frac{2}{x^2}}{4 + \frac{1}{x^3}} = \boxed{\frac{1}{4}}$$

10. $\lim_{x\to\infty} \frac{x^3+1}{x^7+2x^{\frac{7}{2}}} = \lim_{x\to\infty} \frac{x^3+1}{x^7+2x^{\frac{7}{2}}} \cdot \frac{\left(\frac{1}{x^7}\right)}{\left(\frac{1}{x^7}\right)} = \lim_{x\to\infty} \frac{\frac{1}{x^4}+\frac{1}{x^7}}{1+\frac{2}{x^{\frac{7}{2}}}} = \boxed{0}$

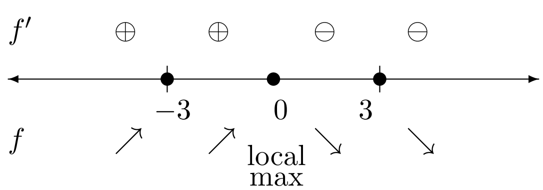
11. $\lim_{x\to\infty} \frac{x^6+1}{x^3+9x^2+7} = \lim_{x\to\infty} \frac{x^6+1}{x^3+9x^2+7} \cdot \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)} = \lim_{x\to\infty} \frac{x^3+\frac{1}{x^3}}{1+\frac{9}{x}+\frac{7}{x^3}} = \boxed{\infty}$

Curve Sketching For each of the following functions, discuss domain, vertical and horizontal asymptotes, intervals of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

12. $f(x) = \frac{1}{x^2-9}$

- Domain: $f(x)$ has domain $\{x | x \neq \pm 3\}$
- VA: Vertical asymptotes at $x = \pm 3$.
- HA: Horizontal asymptote at $y = 0$ for this f since $\lim_{x\to\pm\infty} f(x) = 0$.
- First Derivative Information:

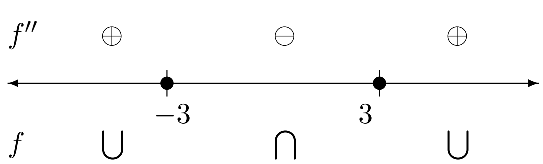
We compute $f'(x) = \frac{-2x}{(x^2-9)^2}$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined or zero. The latter happens when $x = 0$. The derivative is undefined when $x = \pm 3$, but those values are not in the domain of the original function. As a result, $x = 0$ is the critical number. Using sign testing/analysis for f',



So f is increasing on $(-\infty, -3)$ and $(-3, 0)$; and f is decreasing on $(0, 3)$ and $(3, \infty)$. Moreover, f has a local max at $x = 0$ with $f(0) = -\frac{1}{9}$.

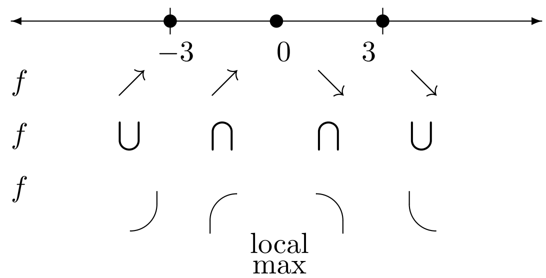
- Second Derivative Information:

Meanwhile, $f'' = \frac{6(x^2+3)}{(x^2-9)^3}$ is never zero. Using sign testing/analysis for f'' around the vertical asymptotes,

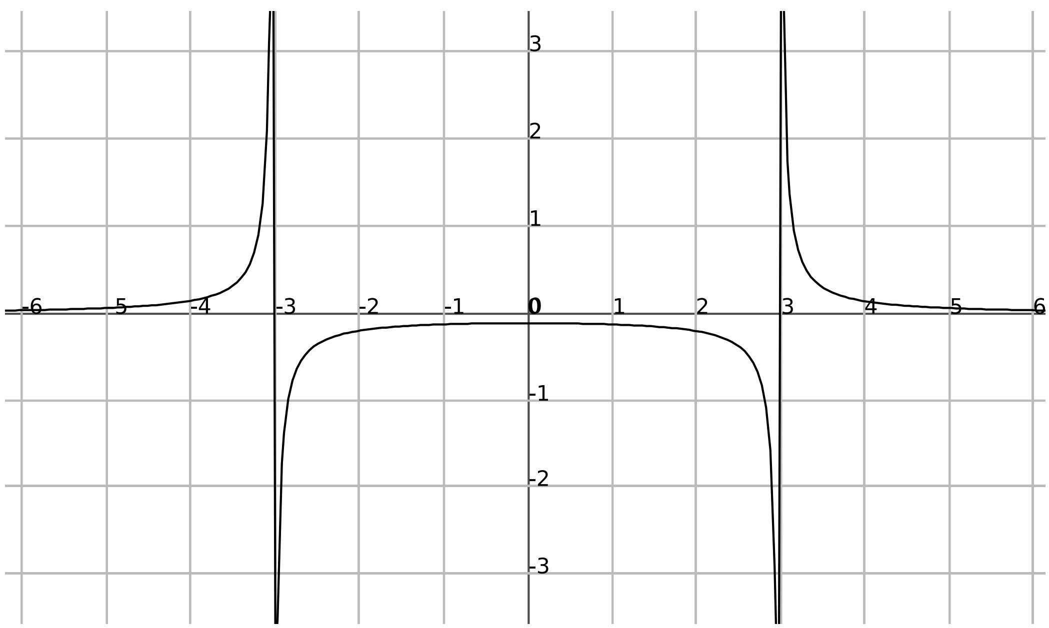


So f is concave down on $(-3,3)$ and concave up on $(-\infty,-3)$ and $(3,\infty)$ with no inflection points.

- Piece the first and second derivative information together:



- Sketch:



13. $f(x) = \dfrac{x^3 + 6x^2 - 40}{(x+2)^3}$.

Take my word for it that (you do NOT have to compute these)

$$f'(x) = \frac{24(x+5)}{(x+2)^4} \text{ and } f''(x) = \frac{-72(x+6)}{(x+2)^5}.$$

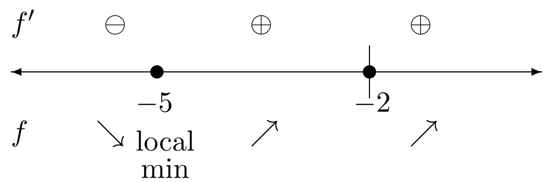
- Domain: $f(x)$ has domain $\{x | x \neq -2\}$

- VA: Vertical asymptotes $x = -2$.
- HA: Horizontal asymptote is $y = 1$ for this f since $\lim_{x\to\pm\infty} f(x) = 1$ because

$$\lim_{x\to\pm\infty} \frac{x^3+6x^2-40}{(x+2)^3} = \lim_{x\to\pm\infty} \frac{x^3+6x^2-40}{x^3+6x^2+12x+8}$$
$$= \lim_{x\to\pm\infty} \frac{x^3+6x^2-40}{x^3+6x^2+12x+8} \cdot \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)} = \lim_{x\to\pm\infty} \frac{1+\frac{6}{x}-\frac{40}{x^3}}{1+\frac{6}{x}+\frac{12}{x^2}+\frac{8}{x^3}} = 1$$

- First Derivative Information:

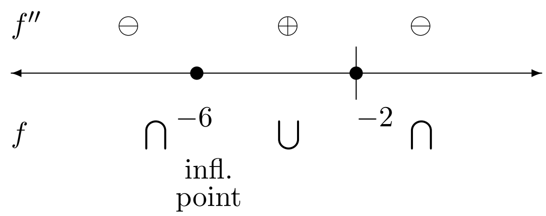
We know $f'(x) = \dfrac{24(x+5)}{(x+2)^4}$. The critical points occur where f' is undefined or zero. The former happens when $x = -2$, but $x = -2$ was not in the domain of the original function, so it isn't technically a critical number. The latter happens when $x = -5$. As a result, $x = -5$ is the critical number. Using sign testing/analysis for f',



So f is decreasing on $(-\infty, -5)$ and increasing on $(-5, -2)$ and $(-2, \infty)$. Moreover, f has a local min at $x = -5$ with $f(-5) = -\dfrac{5}{9}$.

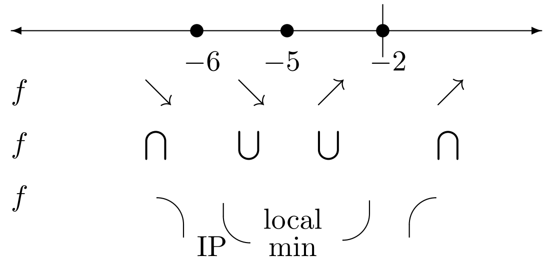
- Second Derivative Information:

Meanwhile, $f'' = \dfrac{-72(x+6)}{(x+2)^5}$. $f'' = 0$ when $x = -6$. Using sign testing/analysis for f'',

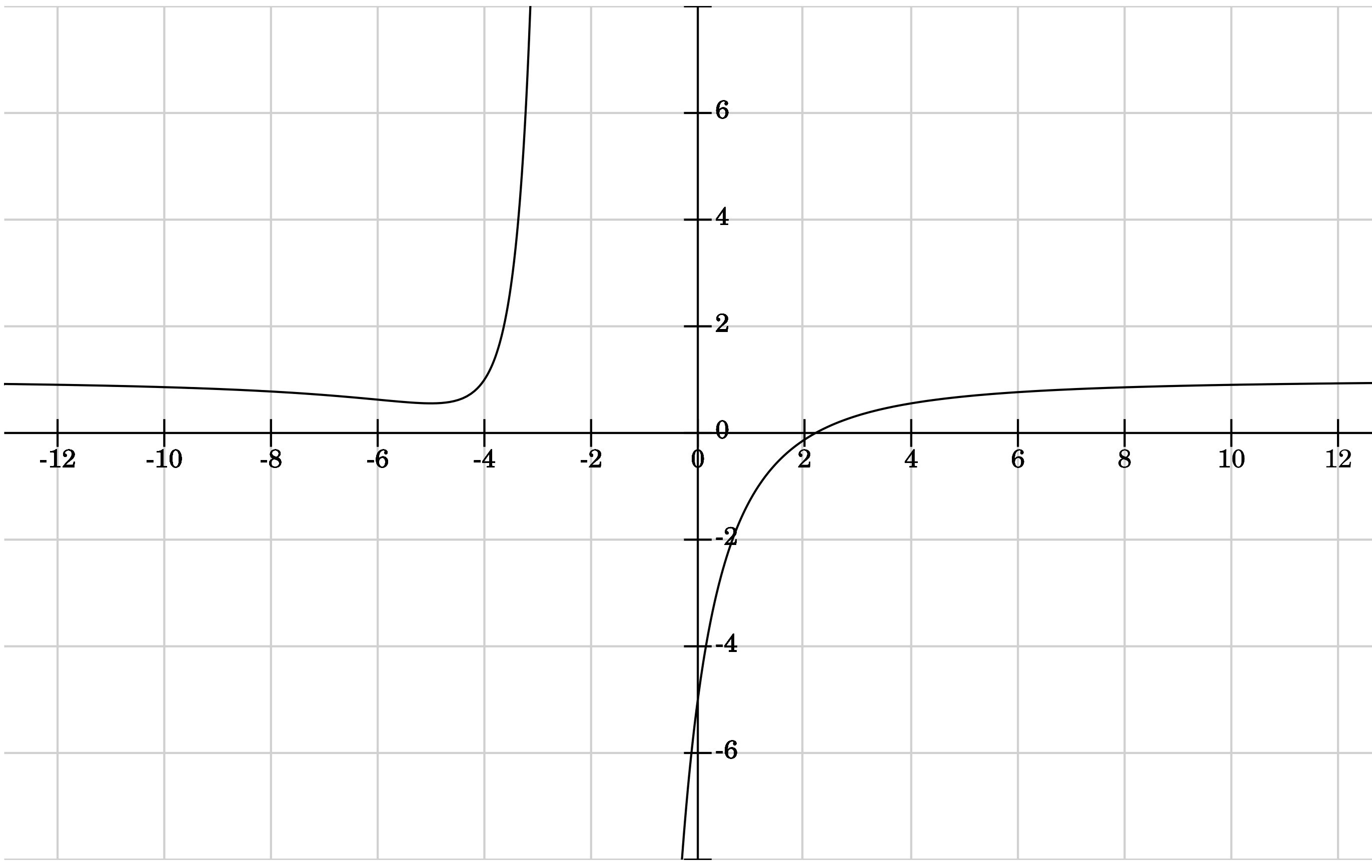


So f is concave down on $(-\infty, -6)$ and $(-2, \infty)$, and concave up on $(-6, -2)$. There is an inflection point at $(-6, \frac{5}{8})$.

- Piece the first and second derivative information together:



- Sketch:



Position, Velocity, Acceleration

14. Suppose that Dan throws a ball, from the ground, straight upward in the air with an initial velocity of 128 meters per second. The ball reaches a height of $\mathbf{s(t) = 128t - 16t^2}$ feet in t seconds. Suppose Sam is lying on the ground under the ball. Answer the following questions:

    (a) What is the maximum height the ball reaches? Max height occurs when $v(t) = 128 - 32t = 0$ or when $t = 4$ seconds. So Max height is $s(4) = 128(4) - 16(4)^2 = 256$ feet.

    (b) What is the ball's velocity at time $t = 5$?
    $v(5) = 128 - 32(5) = -32$ feet per second.

    (c) What is the ball's acceleration at time $t = 5$?
    Note acceleration is constant at $a(t) = -32$ feet per second^2. So $a(5) = -32$ feet per second^2.

    (d) At what time will the ball hit Sam on the ground?
    The ball hits Sam (on the ground) when $s(t) = 0$. That is when $128t - 16t^2 = 16t(8-t) = 0$ or when $t = 0$(start) or $t = 8$ (impact). So the ball hits Sam at $t = 8$ seconds.

    (e) What is the ball's velocity when it hits Sam?
    $v(8) = 128 - 32(8) = 128 - 256 = -128$ feet per second. Think about why the impact velocity is equal in value but opposite in sign to the initial velocty... the graph of $s(t)$ is

a parabola and the slopes on either side of the maximum are equal in value but opposite in sign. (Of course, we are ignoring air resistance here.)

(f) What is the ball's acceleration when it hits Sam?

The ball hits Sam with constant $a(8) = -32$ feet per second^2.

15. RETALIATION! When Dan saw that the ball actually hit Sam, he ran away, up a tree. Dan climbed up the tree exactly 155 feet (above the ground). Revenge was necessary! Sam managed to throw the ball upward at Dan with an initial velocity of 96 feet per second. This time the ball reaches a height of $\mathbf{s(t) = 96t - 16t^2}$ feet in t seconds.

    Does the ball hit Dan? If it doesn't, explain why. If it does, explain why. Show your work.

    Note that max height occurs when $v(t) = 96 - 32t = 0$ or when $t = 3$ seconds. Finally, max height is $s(3) = 96(3) - 16(9) = 288 - 144 = 144$ feet. Therefore, the ball does not hit Dan because the max height is less then Dan's height of 155 feet in the tree.