Math 105 **Practice Exam #3 EXTRA** **Fall 2013** Answer Key

Critical Numbers

1. Find critical numbers for the function $f(x) = \dfrac{x^2+1}{x-3}$.

$f'(x) = \dfrac{(x-3)(2x)-(x^2+1)(1)}{(x-3)^2} = \dfrac{2x^2-6x-x^2-1}{(x-3)^2} = \dfrac{x^2-6x-1}{(x-3)^2}$

Here $f'(x) = 0$ when $x^2 - 6x - 1 = 0$ or using the Quadratic Formula:

$x = \dfrac{6 \pm \sqrt{36-4(1)(-1)}}{2} = \dfrac{6 \pm \sqrt{40}}{2} = \dfrac{6 \pm \sqrt{4}\sqrt{10}}{2} = \dfrac{6 \pm 2\sqrt{10}}{2} = 3 \pm \sqrt{10}$

Notice that $f'(x)$ is undefined at $x = 3$, but $x = 3$ was **not** in the domain of the original function, so it's not technically a critical number.

Finally the critical numbers are $\boxed{x = 3 \pm \sqrt{10}}$

2. Find critical numbers for the function $f(x) = 3x^{\frac{2}{3}} - \dfrac{x}{4}$.

We compute $f'(x) = 2x^{-\frac{1}{3}} - \dfrac{1}{4} = \dfrac{2}{x^{\frac{1}{3}}} - \dfrac{1}{4} = \left(\dfrac{4}{4}\right)\dfrac{2}{x^{\frac{1}{3}}} - \dfrac{1}{4}\left(\dfrac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right) = \dfrac{8}{4x^{\frac{1}{3}}} - \dfrac{x^{\frac{1}{3}}}{4x^{\frac{1}{3}}} = \dfrac{8-x^{\frac{1}{3}}}{x^{\frac{1}{3}}}$.

Here $f'(x) = 0$ when $8 - x^{\frac{1}{3}} = 0$ or when $x^{\frac{1}{3}} = 8$ so when $x = 8^3 = 512$.

f' is undefined at $x = 0$, so $x = 0$ is a critical point since it **is** in the domain of the original function.

Finally the critical numbers are $\boxed{x = 0}$ and $\boxed{x = 512}$.

Absolute Extreme Values

3. Find the absolute maximum and absolute minimum values of

$$F(x) = \frac{1}{x^2-16} \quad \text{on} \quad [-1, 3].$$

$F'(x) = \dfrac{-2x}{x^2-16}$

First $F'(x) = 0$ when $x = 0$. Next $F'(x)$ is undefined when $x = \pm 4$, but $x = \pm 4$ are **not** in the domain of the original function, so they are not technically critical numbers. Hence the only crticial number here is $x = 0$.

Using the Closed Interval Method:

$F(0) = \boxed{-\dfrac{1}{16}} \longleftarrow$ Absolute Maximum Value

$F(-1) = -\dfrac{1}{15}$

$F(3) = \boxed{-\dfrac{1}{7}} \longleftarrow$ Absolute Minimum Value

So the absolute maximum value is $-\dfrac{1}{16}$ (attained at $x = 0$), and the absolute minimum value is $-\dfrac{1}{7}$ (attained at $x = 3$).

4. Find the absolute maximum and absolute minimum values of

$$G(x) = (x-3)^2(x+2)^3 \quad \text{on} \quad [0,4].$$

$G'(x) = (x-3)^2 \cdot 3(x+2)^2 + (x+2)^3 \cdot 2(x-3)$
$= (x-3)(x+2)^2[3(x-3) + 2(x+2)] = (x-3)(x+2)^2[5x-5]$.
On the interval $[0,4]$, G' is always defined. Also, $G'(x) = 0$ happens only when $x = 3$, $x = -2$, and $x = 1$ (our critical numbers). Here $x = -2$ is outside of our interval of interest. Applying the closed interval method:

$G(1) = 108$

$G(3) = \boxed{0} \longleftarrow$ Absolute Minimum Value

$G(0) = 72$

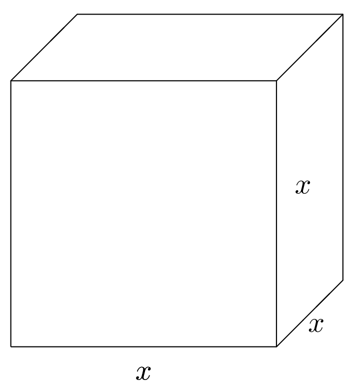
$G(4) = \boxed{216} \longleftarrow$ Absolute Maximum Value

So the absolute maximum value is 216 (attained at $x = 4$), and the absolute minimum value is 0 (attained at $x = 3$).

Related Rates

5. Suppose that garbage is being compacted in the shape of a cube, so that the volume of the cube is shrinking at 6 cubic feet per second. How fast is the length of the edge of the cube changing when the length of the edge has been compacted to 2 feet?

• Diagram



The picture at arbitrary time t is:

• Variables

Let x = length of edge of cube at time t

Let V = Volume of cube at time t

Find $\dfrac{dx}{dt}$ =? when $x = 2$ sec

and $\dfrac{dV}{dt} = -6\dfrac{\text{ft}^3}{\text{second}}$

• Equation relating the variables:

$V = x^3$

• Differentiate both sides w.r.t. time t.

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

• Substitute Key Moment Information (now and not before now!!!):

$\frac{dV}{dt} = 3x^2\frac{dx}{dt}$

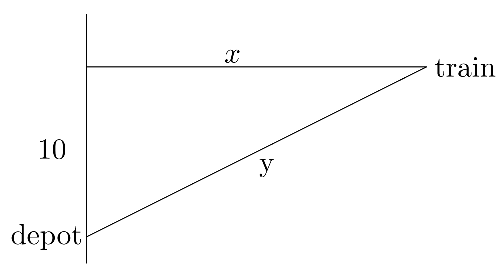
$-6 = 3(2)^2\frac{dx}{dt}$

• Solve for the desired quantity: $\frac{dx}{dt} = -\frac{6}{12} = -\frac{1}{2}\frac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The length of the edge of the cube is shrinking $\frac{1}{2}$ feet every second at that moment.

6. Suppose a train depot is 10 feet directly south from the train track. The train is travelling east at 15 feet per second. How fast is the distance between the train and train depot station increasing when 2 seconds has passed since the train passed directly in front of the depot?

• Diagram



The picture at arbitrary time t is:

• Variables

Let x = distance train has travelled horizontally(east) at time t

Find $\frac{dy}{dt}$ =? when $t = 2$ sec

and $\frac{dx}{dt} = 15\frac{\text{ft}}{\text{second}}$

• Equation relating the variables:

Pythagorean Theorem yields

$x^2 + (10)^2 = y^2$

• Differentiate both sides w.r.t. time t.

$2x\frac{dx}{dt} + 0 = 2y\frac{dy}{dt}$

• Extra Solvable Information: When $t = 2$ seconds, the train has tavelled 30 feet, since its rate is 15 ft/sec.

Then using the Pyth. Thrm, we solve $y = \sqrt{x^2 + 100} = \sqrt{(30)^2 + 100} = \sqrt{900 + 100} = \sqrt{1000} = 10\sqrt{10}$

• Substitute Key Moment Information (now and not before now!!!):

$2x\dfrac{dx}{dt} = 2y\dfrac{dy}{dt}$

$2(30)(15) = 2(10\sqrt{10})\dfrac{dy}{dt}$

• Solve for the desired quantity:

$\dfrac{dy}{dt} = \dfrac{2(30)(15)}{2(10)\sqrt{10}} = \dfrac{45}{\sqrt{10}}\dfrac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The distance between the train and the depot is increasing $\dfrac{45}{\sqrt{10}}$ feet every second.

Limits at Infinity

7. Compute each of the following limits at infinity. Please show your work.

(a) $\displaystyle\lim_{x\to\infty} \frac{15-3x^2}{7x^2+20} = \lim_{x\to\infty} \frac{15-3x^2}{7x^2+20} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$

$\displaystyle = \lim_{x\to\infty} \frac{\frac{15}{x^2} - 3}{7 + \frac{20}{x^2}} = \boxed{-\frac{3}{7}}$

(b) $\displaystyle\lim_{x\to\infty} \frac{x^9+8x^7+6x^5+4}{3x^2+1} = \lim_{x\to\infty} \frac{x^9+8x^7+6x^5+4}{3x^2+1} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$

$\displaystyle = \lim_{x\to\infty} \frac{x^7+8x^5+6x^3+\frac{4}{x^2}}{3+\frac{1}{x^2}} = \boxed{\infty}$

(c) $\displaystyle\lim_{x\to\infty} \frac{6x^7-7x^6+2}{7x^8+8x^7+2} = \lim_{x\to\infty} \frac{6x^7-7x^6+2}{7x^8+8x^7+2} \cdot \frac{\left(\frac{1}{x^8}\right)}{\left(\frac{1}{x^8}\right)}$

$\displaystyle = \lim_{x\to\infty} \frac{\frac{6}{x} - \frac{7}{x^2} + \frac{2}{x^8}}{7 + \frac{8}{x} + \frac{2}{x^8}} = \boxed{0}$

Curve Sketching For each of the following functions, discuss domain, vertical and horizontal asymptotes, intervals of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

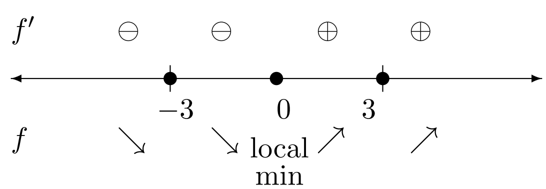
8. $f(x) = \dfrac{x^2-16}{x^2-9}$

- $f(x)$ has domain $\{x|x \neq \pm 3\}$
- Vertical asymptotes at $x = \pm 3$.
- Horizontal asymptote at $y = 1$ for this f since

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2-16}{x^2-9} = \lim_{x\to\pm\infty} \frac{x^2-16}{x^2-9} \cdot \left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}} \right) = \lim_{x\to\pm\infty} \frac{1-\frac{16}{x^2}}{1-\frac{9}{x^2}} = 1.$$

- First Derivative Information

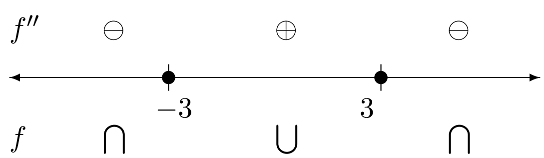
We use $f'(x) = \dfrac{14x}{(x^2-9)^2}$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined or zero. The latter happens when $x = 0$. The derivative is undefined when $x = \pm 3$, but those values are not in the domain of the original function. As a result, $x = 0$ is the critical number. Using sign testing/analysis for f',



So f is decreasing on $(-\infty, -3)$ and $(-3, 0)$; and f is increasing on $(0, 3)$ and $(3, \infty)$. Moreover, f has a local min at $x = 0$.

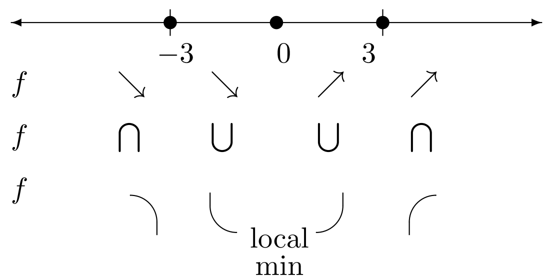
- Second Derivative Information

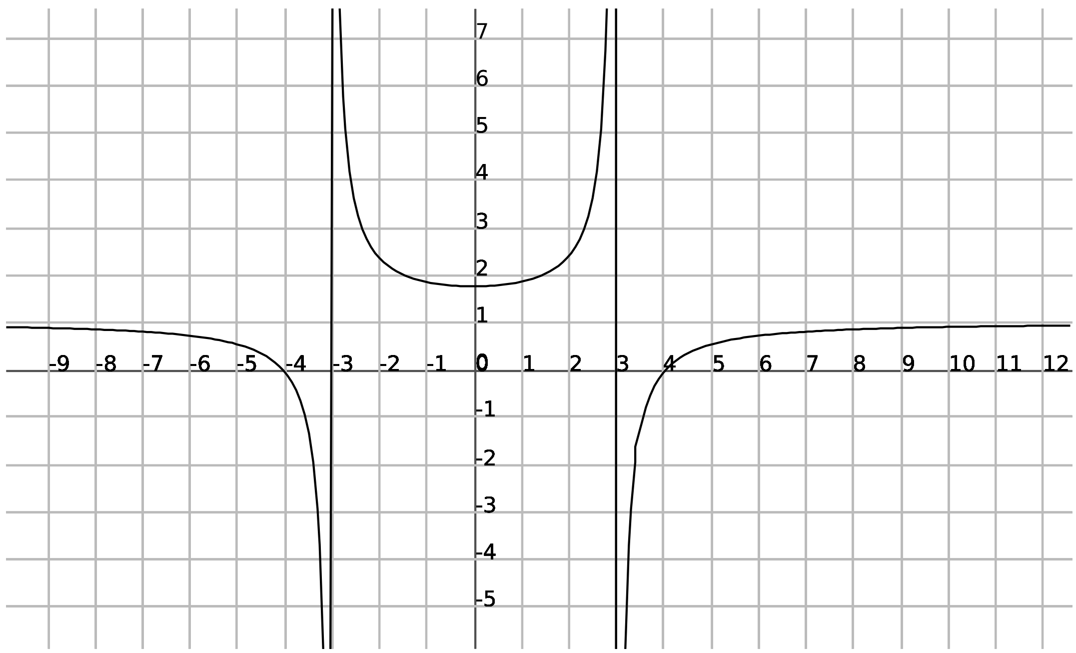
Meanwhile, $f'' = \dfrac{-42(x^2+3)}{(x^2-9)^3}$ is never zero. Using sign testing/analysis for f'' around the vertical asymptotes,



So f is concave up on $(-3, 3)$ and concave down on $(-\infty, -3)$ and $(3, \infty)$.

• Piece the first and second derivative information together





9. $f(x) = \dfrac{-x^2+x+2}{x^2-2x+1}$. Take my word for it that (you do NOT have to compute these)

$$f'(x) = \frac{x-5}{(x-1)^3} \text{ and } f''(x) = \frac{-2x+14}{(x-1)^4}.$$

• Domain: $f(x)$ has domain $\{x|x \neq 1\}$

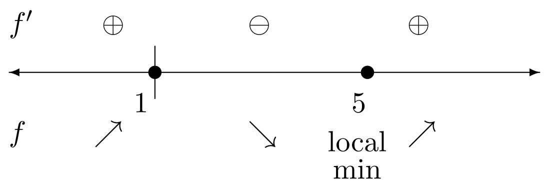
• VA: Vertical asymptotes $x = 1$.

• HA: Horizontal asymptote is $y = -1$ for this f since $\lim\limits_{x\to\pm\infty} f(x) = -1$ because

$$\lim_{x\to\pm\infty} \frac{-x^2+x+2}{x^2-2x+1} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\pm\infty} \frac{-1+\frac{1}{x}+\frac{2}{x^2}}{1-\frac{2}{x}+\frac{1}{x^2}} = -1$$

• First Derivative Information:

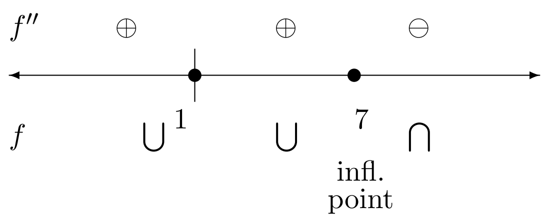
We know $f'(x) = \dfrac{x-5}{(x-1)^3}$. The critical points occur where f' is undefined or zero. The former happens when $x = 1$, but $x = 1$ was not in the domain of the original function, so it isn't technically a critical number. The latter happens when $x = 5$. As a result, $x = 5$ is the critical number. Using sign testing/analysis for f',



So f is decreasing on $(1, 5)$ and increasing on $(-\infty, 1)$ and $(5, \infty)$. Moreover, f has a local mi n at $x = 5$ with $f(5) = -\dfrac{9}{8}$.

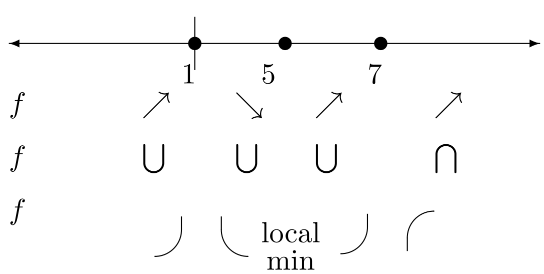
• Second Derivative Information:

Meanwhile, $f'' = \dfrac{-2x+14}{(x-1)^4}$. $f'' = 0$ when $x = 7$. Using sign testing/analysis for f'',

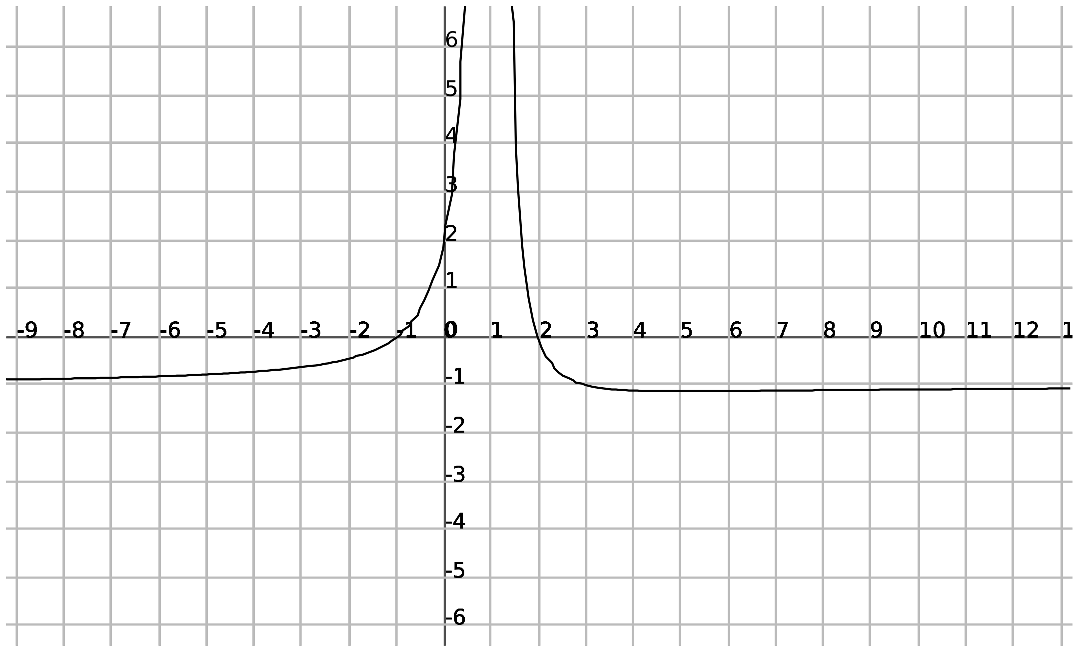


So f is concave down on $(7, \infty))$ and concave up on $(-\infty, 1)$ and $(1, 7)$. There is an inflection point at $(7, -\frac{10}{9})$.

• Piece the first and second derivative information together:



- Sketch:



Position, Velocity, Acceleration

10. A ball is thrown straight upward from the ground with initial velocity $v_0 = 128$ feet per second. The height of the ball at time t is given by the position function $s(t) = -16t^2 + 128t$ feet in t seconds.

    Answer the following questions:

    (a) What is the maximum height attained by the ball?

    Maximum height is attained when $v(t) = -32t + 128 = 0$ so $t = \dfrac{128}{32} = 4$ seconds.

    The Maximum height is $s(4) = -16(4)^2 + 128(4) = -256 + 512 = 256$ feet.

    (b) Find the velocity with which the ball hits the ground upon its return, at impact. The ball hits the ground when $s(t) = -16t^2 + 128t = 0$. This equation factors as

    $s(t) = -16t(t-8) = 0$ and $t = 0$ (launch) or $t = 8$ (impact). The ball hits the ground after 8 seconds. The velocity at impact is $v(8) = -32(8) + 128 = -256 + 128 = -128$ feet per second.