Math 105 **Practice Exam #3** **Fall 2013** Answer Key

Critical Numbers

1. Find critical numbers for the function $f(x) = x^{\frac{1}{3}}(8-x)$.

   First compute the derivative

   $f'(x) = x^{\frac{1}{3}}(-1) + (8-x)\dfrac{1}{3}x^{-\frac{2}{3}} = x^{\frac{1}{3}}(-1) + \dfrac{8-x}{3x^{\frac{2}{3}}} = \left(\dfrac{3x^{\frac{2}{3}}}{3x^{\frac{2}{3}}}\right)x^{\frac{1}{3}}(-1) + \dfrac{8-x}{3x^{\frac{2}{3}}}$

   $= \left(\dfrac{-3x}{3x^{\frac{2}{3}}}\right) + \dfrac{8-x}{3x^{\frac{2}{3}}} = \dfrac{-3x+8-x}{3x^{\frac{2}{3}}} = \dfrac{-4x+8}{3x^{\frac{2}{3}}}$

   Critical numbers are where the derivative equals 0 or is undefined.

   First $f'(x)$ equals zero when the numerator equals 0, which is when $x = 2$.

   Second the derivative is undefined here where the denominator is 0, which is when $x = 0$, which we should note **is** in the domain of the original function.

   Finally the critical numbers are $\boxed{x = 2}$ and $\boxed{x = 0}$.

2. Find critical numbers for the function $f(x) = \dfrac{2x^3 + x^2 - 1}{x^3}$.

   First compute the derivative

   $f'(x) = \dfrac{x^3(6x^2+2x) - (2x^3+x^2-1)(3x^2)}{x^6} = \dfrac{6x^5 + 2x^4 - 6x^5 - 3x^4 + 3x^2}{x^6}$

   $= \dfrac{-x^4 + 3x^2}{x^6} = \dfrac{x^2(-x^2+3)}{x^6} = \dfrac{-x^2+3}{x^4}$

   Critical numbers are where the derivative equals 0 or is undefined.

   First $f'(x)$ equals zero when the numerator equals 0, which is when $-x^2 + 3 = 0$ or when $x = \pm\sqrt{3}$.

   Second the derivative is undefined here where the denominator is 0, which is when $x = 0$, which we should note is **not** in the domain of the original function.

   Finally the only critical numbers $\boxed{x = \pm\sqrt{3}}$.

Absolute Extreme Values

3. Find the absolute maximum and absolute minimum values of

   $$F(x) = x\sqrt{4-x^2} \quad \text{on} \quad [-1, 2].$$

   First compute the derivative

   $f'(x) = x\dfrac{1}{2\sqrt{4-x^2}}(-2x) + \sqrt{4-x^2}(1) = \dfrac{-2x^2}{2\sqrt{4-x^2}} + \sqrt{4-x^2}\left(\dfrac{2\sqrt{4-x^2}}{2\sqrt{4-x^2}}\right)$

$= \dfrac{-2x^2}{2\sqrt{4-x^2}} + \left(\dfrac{2(4-x^2)}{2\sqrt{4-x^2}}\right) = \dfrac{-2x^2+8-2x^2}{2\sqrt{4-x^2}} = \dfrac{8-4x^2}{2\sqrt{4-x^2}}.$

$f'(x) \overset{\text{set}}{=} 0$ when $8-4x^2 = 0$ or $x = \pm\sqrt{2}$.

$f'(x)$ is undefined at $x = \pm 2$, which **are** in the domain of the original function.

So the critical numbers are $x = \pm 2$ and $x = \pm\sqrt{2}$. Here $x = -2$ and $x = -\sqrt{2}$ are not in the interval of interest.

Applying the Closed Interval method:

$f(\sqrt{2}) = \sqrt{2}\sqrt{2} = \boxed{2} \longleftarrow$ Absolute Maximum Value

$f(-1) = \boxed{-\sqrt{3}} \longleftarrow$ Absolute Minimum Value

$f(2) = 0$

So the absolute maximum value is 2 (attained at $x = \sqrt{2}$), and the absolute minimum value is $-\sqrt{3}$ (attained at $x = -1$).

4. Find the absolute maximum and absolute minimum values of

$$G(x) = x^3 + 6x^2 - 1 \quad \text{on} \quad [-1, 1].$$

$G'(x) = 3x^2 + 12x = 3x(x+4)$ Simplify fully.

$G'(x) \overset{\text{set}}{=} 0$ when $x = 0$ or $x = -4$.

$G'(x)$ is always defined here, since it's a polynomial.

So the critical numbers are $x = 0$ and $x = -4$, but $x = -4$ is NOT in the interval of interest here.

Applying the Closed Interval method:

$G(0) = \boxed{-1} \longleftarrow$ Absolute Minimum Value

$G(1) = \boxed{6} \longleftarrow$ Absolute Maximum Value

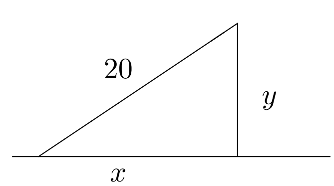
$G(-1) = 4$

So the absolute maximum value is 6 (attained at $x = 1$), and the absolute minimum value is -1 (attained at $x = 0$).

Related Rates

5. Suppose a 20 foot ladder is sliding down a vertical wall. The base of the ladder is sliding on the level ground, away from the wall, at 2 feet per second. At what rate is the top of the ladder sliding down after 5 seconds has passed?

- Diagram



• Variables
Let x = distance between bottom of ladder and wall at time t
Let y = distance between top of ladder and ground at time t
Find $\frac{dy}{dt}$ =? when $t = 5$ seconds
and $\frac{dx}{dt} = 2\frac{\text{ft}}{\text{sec}}$

• Equation relating the variables:

We have $x^2 + y^2 = (20)^2$.

• Differentiate both sides w.r.t. time t.

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ (Related Rates!)

• Extra solvable information: If 5 seconds have passed, and the ladder slides at a rate of 2 feet every second, then the ladder has moved $x = 10$ feet on the ground.

Then if $x = 10$ at the key moment and the hypotenuse is fixed at 20 then using the Pythagorean Theorem, we have $y = \sqrt{400 - 100} = \sqrt{300} = 10\sqrt{3}$ at that key moment.

• Substitute Key Moment Information (now and not before now!!!):

$2(10)(2) + 2(10)\sqrt{3}\frac{dy}{dt} = 0$

• Solve for the desired quantity:

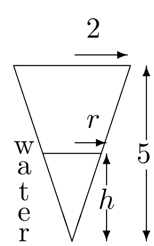
$\frac{dy}{dt} = -\frac{40}{20\sqrt{3}} = -\frac{2}{\sqrt{3}}\frac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The top of the ladder is sliding down the wall at a rate of $\frac{2}{\sqrt{3}}$ feet every second.

6. A conical paper cup of water is 4 inches across the top and 5 inches deep. It has a hole in the bottom point and is leaking water at 2 inches^3 per second. At what rate is the height of the water decreasing when the water height is 1 inch?

The cross section (with water level drawn in) looks like:

• Diagram



• Variables
Let r = radius of the water level at time t
Let h = height of the water level at time t
Let V = volume of the water in the tank at time t
Find $\frac{dh}{dt}$ =? when $h = 1$ feet
and $\frac{dV}{dt} = -2\frac{\text{in}^3}{\text{sec}}$

• Equation relating the variables:

Volume= $V = \frac{1}{3}\pi r^2 h$

• Extra solvable information: Note that r is not mentioned in the problem's info. But there is a relationship, via similar triangles, between r and h. We must have

$$\frac{r}{2} = \frac{h}{5} \implies r = \frac{2h}{5}$$

After substituting into our previous equation, we get:

$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h = \frac{4}{75}\pi h^3$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{75}\pi h^3\right) \implies \frac{dV}{dt} = \frac{4}{75}\pi \cdot 3h^2 \cdot \frac{dh}{dt} \implies \frac{dV}{dt} = \frac{4}{25}\pi h^2 \frac{dh}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$-2 = \frac{4}{25}\pi(1)^2\frac{dh}{dt}$

• Solve for the desired quantity:

$\frac{dh}{dt} = -\frac{50}{4\pi} = -\frac{25}{2\pi}\frac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The water height is decreasing at a rate of $\frac{25}{2\pi}$ inches every second at that moment.

Limits at Infinity

7. Compute each of the following limits at infinity. Please show your work.

(a) $\lim\limits_{x\to\infty} \frac{x^3 - 5x^2 - 90}{-9x^3 - 6x^2 + 4} = \lim\limits_{x\to\infty} \frac{x^3 - 5x^2 - 90}{-9x^3 - 6x^2 + 4} \cdot \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)}$

$= \lim\limits_{x\to\infty} \frac{1 - \frac{5}{x} - \frac{90}{x^3}}{-9 - \frac{6}{x} + \frac{4}{x^3}} = \boxed{-\frac{1}{9}}$

(b) $\lim\limits_{x\to\infty} \frac{x^2 - x + 1}{2x^5 + 7x^2 + 3} = \lim\limits_{x\to\infty} \frac{x^2 - x + 1}{2x^5 + 7x^2 + 3} \cdot \frac{\left(\frac{1}{x^5}\right)}{\left(\frac{1}{x^5}\right)} = \lim\limits_{x\to\infty} \frac{\frac{1}{x^3} - \frac{1}{x^4} + \frac{1}{x^5}}{2 + \frac{7}{x^3} + \frac{3}{x^5}} = \boxed{0}$

(c) $\lim\limits_{x\to\infty} \frac{x^{99} + 99}{100x^{98} + x + 97} = \lim\limits_{x\to\infty} \frac{x^{99} + 99}{100x^{98} + x + 97} \cdot \frac{\left(\frac{1}{x^{98}}\right)}{\left(\frac{1}{x^{98}}\right)}$

$= \lim\limits_{x\to\infty} \frac{x + \frac{99}{x^{98}}}{100 + \frac{1}{x^{97}} + \frac{97}{x^{98}}} = \boxed{\infty}$

Curve Sketching For each of the following functions, discuss domain, vertical and horizontal asymptotes, intervals of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

8. $f(x) = x^4 - 6x^2$

• Domain: $f(x)$ has domain $(-\infty, \infty)$

• VA: It is a polynomial, continuous everywhere, and so has no vertical asymptotes.

• HA: There are no horizontal asymptotes for this f since $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$ because $\lim_{x\to\infty} x^4 - 6x^2 = x^2(x^2 - 6) = \infty$ and $\lim_{x\to-\infty} x^4 - 6x^2 = x^2(x^2 - 6) = \infty$

• First Derivative Information:

We compute $f'(x) = 4x^3 - 12x$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined (never here) or zero. The latter happens when

$$4x^3 - 12x = 4x(x^2 - 3) = 0 \Longrightarrow x = 0 \text{ or } x = \pm\sqrt{3}$$

As a result, $x = 0$ and $x = \pm\sqrt{3}$ are the critical numbers. Using sign testing/analysis for f',

f'	$\ominus$		$\oplus$		$\ominus$		$\oplus$
		$-\sqrt{3}$		0		$\sqrt{3}$	
f	$\searrow$	local min	$\nearrow$	local max	$\searrow$	local min	$\nearrow$

So f is increasing on $(-\sqrt{3}, 0)$and $(\sqrt{3}, \infty)$; and f is decreasing on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$. Moreover, f has a local max at $x = 0$ with $f(0) = 0$, and local mins at $x = \pm\sqrt{3}$ with $f(\pm\sqrt{3}) = -9$.

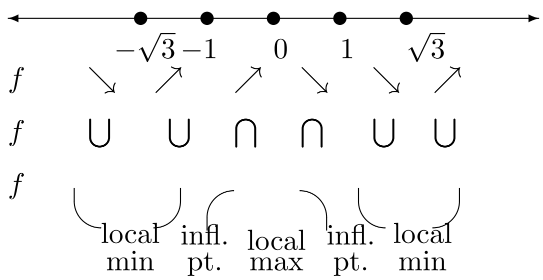
• Second Derivative Information:

Meanwhile, f'' is always defined and continuous, and $f'' = 12x^2 - 12 = 0$ only at our possible inflection points $x = \pm 1$. Using sign testing/analysis for f'',

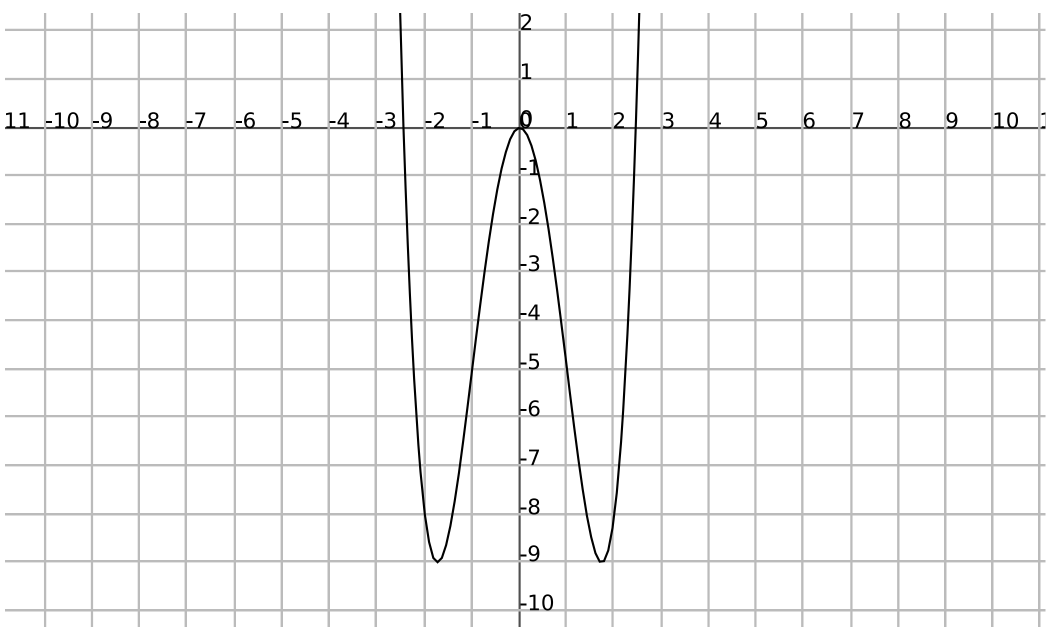
f''	$\oplus$		$\ominus$		$\oplus$
		-1		1	
f	$\cup$	infl. point	$\cap$	infl. point	$\cup$

So f is concave down on $(-1, 1)$ and concave up on $(-\infty, -1)$ and $(1, \infty)$, with an inflection points at $x = \pm 1$ with $f(\pm 1) = -5$.

- Piece the first and second derivative information together:



- Sketch:



9. $f(x) = \dfrac{3x^2}{1-x^2}$. Take my word for it that (you do NOT have to compute these)

$$f'(x) = \frac{6x}{(1-x^2)^2} \text{ and } f''(x) = \frac{6(1+3x^2)}{(1-x^2)^3}.$$

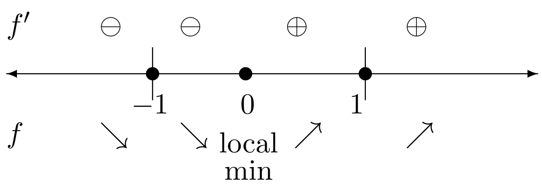
- Domain: $f(x)$ has domain $\{x | x \neq \pm 1\}$
- VA: Vertical asymptotes $x = \pm 1$.

• HA: Horizontal asymptote is $y = -3$ for this f since $\lim_{x \to \pm\infty} f(x) = -3$ because

$$\lim_{x \to \pm\infty} \frac{3x^2}{1 - x^2} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x \to \pm\infty} \frac{3}{\frac{1}{x^2} - 1} = -3$$

• First Derivative Information:

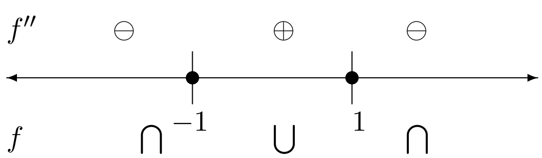
Take $f'(x) = \dfrac{6x}{(1 - x^2)^2}$ and set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined or zero. The former happens when $x = \pm 1$, but $x = \pm 1$ were not in the domain of the original function, so they aren't technically critical numbers. The latter happens when $x = 0$. As a result, $x = 0$ is the critical number. Using sign testing/analysis for f',



So f is decreasing on $(-\infty, -1)$ and $(-1, 0)$ and increasing on $(0, 1)$ and $(1, \infty)$. Moreover, f has a local min at $x = 0$ with $f(0) = 0$.

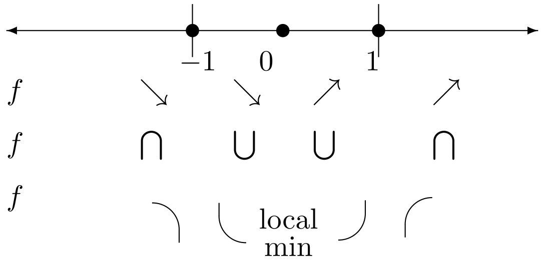
• Second Derivative Information:

Meanwhile, $f'' = \dfrac{6(1 + 3x^2)}{(1 - x^2)^3}$. Using sign testing/analysis for f'',

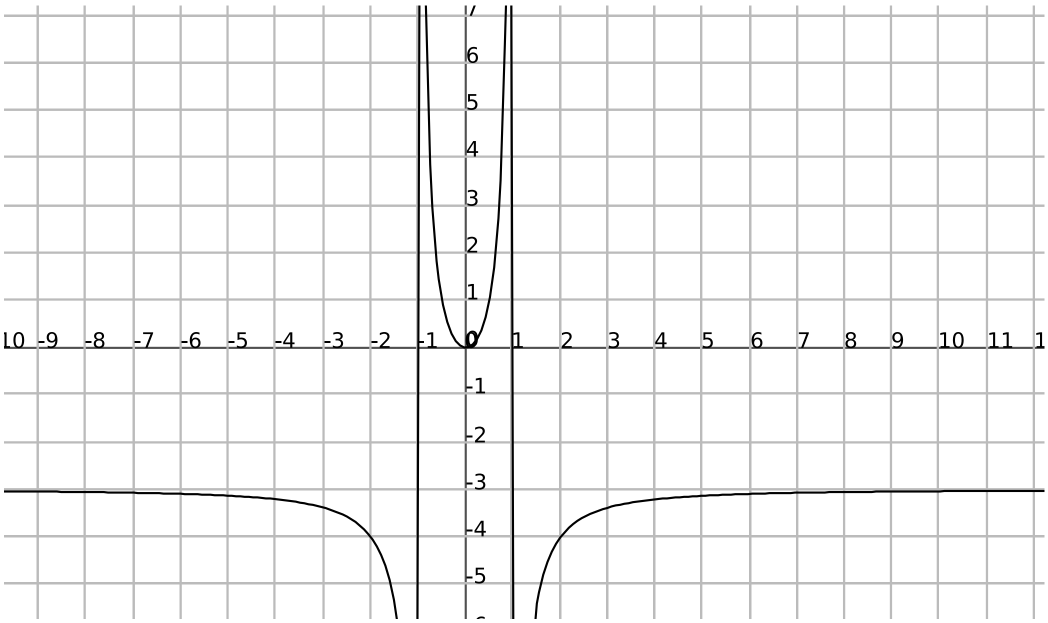


So f is concave down on $(-\infty, -1)$ and$(1, \infty)$ and concave up on $(-1, 1)$.

• Piece the first and second derivative information together:



- Sketch:



Position, Velocity, Acceleration

10. A ball is thrown straight upward from the ground with initial velocity $v_0 = 96$ feet per second. The height of the ball at time t is given by the position function $s(t) = -16t^2 + 96t$ feet in t seconds.

    Answer the following questions:

    (a) What is the maximum height attained?

    Note that max height occurs when $v(t) = 96 - 32t = 0$ or when $t = 3$ seconds.

    Finally, max height is $s(3) = -16(9) + 96(3) = -144 + 288 = 144$ feet.

    (b) Find the velocity with which the ball hits the ground upon its return, at impact.

    Ball hits the ground when $s(t) = -16t^2 + 96t = -16t(t-6) - 0$ or when $t = 0$ (launch) and $t = 6$ (impact). So the ball hits the ground on the return trip when $t = 6$ seconds.

    Velocity at impact is given by $v(6) = 96 - 32(6) = 96 - 192 = -96$ ft/sec.

    (c) How much time has passed before the ball returned to the ground?

    Six seconds have passed before the ball returns to the ground.

    (d) When is the ball 128 feet above the ground?

    The ball is 128 feet above the ground when $s(t) = -16t^2 + 96t = 128$.

    That is when $-16t^2+96t-128 = 0$ which is factors as $-16(t^2-6t+8) = -16(t-2)(t-4) = 0$. So $t = 2$ and $t = 4$ seconds.