Math 105 Midterm Exam #3 December 6, 2013

1. [15 Points] **Critical Numbers**

(a) Find critical numbers for the function $f(x) = \dfrac{x^2+1}{x-3}$.

$$f'(x) = \frac{(x-3)(2x)-(x^2+1)(1)}{(x-3)^2} = \frac{2x^2-6x-x^2-1}{(x-3)^2} = \frac{x^2-6x-1}{(x-3)^2}$$

Here $f'(x) = 0$ when $x^2 - 6x - 1 = 0$ or using the Quadratic Formula:

$$x = \frac{6 \pm \sqrt{36-4(1)(-1)}}{2} = \frac{6\pm\sqrt{40}}{2} = \frac{6\pm\sqrt{4}\sqrt{10}}{2} = \frac{6\pm 2\sqrt{10}}{2} = 3\pm\sqrt{10}$$

Notice that $f'(x)$ is undefined at $x = 3$, but $x = 3$ was **not** in the domain of the original function, so it's not technically a critical number.

Finally the critical numbers are $\boxed{x = 3\pm\sqrt{10}}$

(b) Find the critical numbers for $f(x) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$.

First $f'(x) = \frac{4}{3}x^{\frac{1}{3}} - \left(\frac{4}{3}\right)x^{-\frac{2}{3}} = \frac{4x^{\frac{1}{3}}}{3} - \frac{4}{3x^{\frac{2}{3}}} = \left(\frac{x^{\frac{2}{3}}}{x^{\frac{2}{3}}}\right)\frac{4x^{\frac{1}{3}}}{3} - \frac{4}{3x^{\frac{2}{3}}} = \frac{4x}{3x^{\frac{2}{3}}} - \frac{4}{3x^{\frac{2}{3}}} = \frac{4x-4}{3x^{\frac{2}{3}}} = 0$

when the numerator equals 0, which is when $4x - 4 = 0$ or when $x = 1$.

Secondly the derivative is underfined when the denominator equals 0 here, when $x = 0$, which is in the domain of the original function.

Finally the critical numbers are $\boxed{x = 1}$ and $\boxed{x = 0}$

2. [20 Points] **Absolute Extreme Values**

(a) Find the absolute maximum and absolute minimum values of

$$G(x) = (x-3)^2(x+2)^3 \quad \text{on} \quad [0,4].$$

$G'(x) = (x-3)^2 \cdot 3(x+2)^2 + (x+2)^3 \cdot 2(x-3)$

$= (x-3)(x+2)^2[3(x-3)+2(x+2)] = (x-3)(x+2)^2[5x-5]$.

On the interval $[0,4]$, G' is always defined. Also, $G'(x) = 0$ happens only when $x = 3$, $x = -2$, and $x = 1$ (our critical numbers). Here $x = -2$ is outside of our interval of interest. Applying the closed interval method:

$G(1) = 108$

$G(3) = \boxed{0} \longleftarrow$ Absolute Minimum Value

$G(0) = 72$

$G(4) = \boxed{216} \longleftarrow$ Absolute Maximum Value

So the absolute maximum value is 216 (attained at $x = 4$), and the absolute minimum value is 0 (attained at $x = 3$).

(b) Find the absolute maximum and absolute minimum values of

$$F(x) = x\sqrt{4-x^2} \quad \text{on} \quad [-1, 2].$$

First compute the derivative

$$f'(x) = x\frac{1}{2\sqrt{4-x^2}}(-2x) + \sqrt{4-x^2}(1) = \frac{-2x^2}{2\sqrt{4-x^2}} + \sqrt{4-x^2}\left(\frac{2\sqrt{4-x^2}}{2\sqrt{4-x^2}}\right)$$

$$= \frac{-2x^2}{2\sqrt{4-x^2}} + \left(\frac{2(4-x^2)}{2\sqrt{4-x^2}}\right) = \frac{-2x^2+8-2x^2}{2\sqrt{4-x^2}} = \frac{8-4x^2}{2\sqrt{4-x^2}}.$$

$f'(x) \overset{\text{set}}{=} 0$ when $8 - 4x^2 = 0$ or $x = \pm\sqrt{2}$.

$f'(x)$ is undefined at $x = \pm 2$, which **are** in the domain of the original function.

So the critical numbers are $x = \pm 2$ and $x = \pm\sqrt{2}$. Here $x = -2$ and $x = -\sqrt{2}$ are not in the interval of interest.

Applying the Closed Interval method:

$f(\sqrt{2}) = \sqrt{2}\sqrt{2} = \boxed{2} \longleftarrow$ Absolute Maximum Value

$f(-1) = \boxed{-\sqrt{3}} \longleftarrow$ Absolute Minimum Value

$f(2) = 0$

So the absolute maximum value is 2 (attained at $x = \sqrt{2}$), and the absolute minimum value is $-\sqrt{3}$ (attained at $x = -1$).

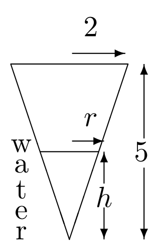
3. [20 Points] **Related Rates**

A conical paper cup of water is 4 inches across the entire top and 5 inches deep. It has a hole in the bottom point and is leaking water at 2 cubic inches per second. At what rate is the height of the water level decreasing when the water height is 1 inch?

*** Recall the volume of the cone is given by $V = \frac{1}{3}\pi r^2 h$***

The cross section (with water level drawn in) looks like:

- Diagram



- Variables

Let r = radius of the water level at time t

Let h = height of the water level at time t

Let V = volume of the water in the tank at time t

Find $\frac{dh}{dt}$ =? when $h = 1$ feet

and $\frac{dV}{dt} = -2\frac{\text{in}^3}{\text{sec}}$

• Equation relating the variables:

Volume= $V = \frac{1}{3}\pi r^2 h$

• Extra solvable information: Note that r is not mentioned in the problem's info. But there is a relationship, via similar triangles, between r and h. We must have

$$\frac{r}{2} = \frac{h}{5} \implies r = \frac{2h}{5}$$

After substituting into our previous equation, we get:

$V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h = \frac{4}{75}\pi h^3$

• Differentiate both sides w.r.t. time t.

$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{4}{75}\pi h^3\right) \implies \frac{dV}{dt} = \frac{4}{75}\pi \cdot 3h^2 \cdot \frac{dh}{dt} \implies \frac{dV}{dt} = \frac{4}{25}\pi h^2 \frac{dh}{dt}$ (Related Rates!)

• Substitute Key Moment Information (now and not before now!!!):

$-2 = \frac{4}{25}\pi(1)^2\frac{dh}{dt}$

• Solve for the desired quantity:

$\frac{dh}{dt} = -\frac{50}{4\pi} = -\frac{25}{2\pi}\frac{\text{ft}}{\text{sec}}$

• Answer the question that was asked: The water height is decreasing at a rate of $\frac{25}{2\pi}$ inches every second at that moment.

4. [15 Points] **Limits at Infinity**

(a) $\lim\limits_{x\to\infty} \frac{x^9 + 8x^7 + 6x^5 + 4}{3x^2+1} = \lim\limits_{x\to\infty} \frac{x^9 + 8x^7 + 6x^5 + 4}{3x^2+1} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$

$= \lim\limits_{x\to\infty} \frac{x^7 + 8x^5 + 6x^3 + \frac{4}{x^2}}{3 + \frac{1}{x^2}} = \boxed{\infty}$

(b) $\lim\limits_{x\to-\infty} \frac{1-x^3}{7x^3+x^2-100} = \lim\limits_{x\to\infty} \frac{1-x^3}{7x^3+x^2-100} \cdot \frac{\left(\frac{1}{x^3}\right)}{\left(\frac{1}{x^3}\right)}$

$= \lim\limits_{x\to\infty} \frac{\frac{1}{x^3} - 1}{7 + \frac{1}{x} - \frac{100}{x^3}} = \boxed{-\frac{1}{7}}$

(c) $\lim_{x\to\infty} \dfrac{x^2-x+1}{2x^5+7x^2+3} = \lim_{x\to\infty} \dfrac{x^2-x+1}{2x^5+7x^2+3} \cdot \dfrac{\left(\dfrac{1}{x^5}\right)}{\left(\dfrac{1}{x^5}\right)} = \lim_{x\to\infty} \dfrac{\dfrac{1}{x^3}-\dfrac{1}{x^4}+\dfrac{1}{x^5}}{2+\dfrac{7}{x^3}+\dfrac{3}{x^5}} = \boxed{0}$

5. [20 Points] **Curve Sketching** Let $f(x) = \dfrac{-x^2+x+2}{x^2-2x+1}$.

For this function, discuss domain, vertical and horizontal asymptotes, intervals of increase or decrease, local extreme value(s), concavity, and inflection point(s). Then use this information to present a detailed and labelled sketch of the curve.

Take my word for it that (you do **NOT** have to compute these)

$$f'(x) = \frac{x-5}{(x-1)^3} \qquad \text{and} \qquad f''(x) = \frac{-2x+14}{(x-1)^4}.$$

• Domain: $f(x)$ has domain $\{x|x \neq 1\}$

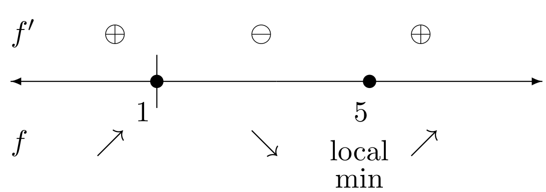
• VA: Vertical asymptotes $x = 1$.

• HA: Horizontal asymptote is $y = -1$ for this f since $\lim_{x\to\pm\infty} f(x) = -1$ because

$$\lim_{x\to\pm\infty} \frac{-x^2+x+2}{x^2-2x+1} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\pm\infty} \frac{-1+\frac{1}{x}+\frac{2}{x^2}}{1-\frac{2}{x}+\frac{1}{x^2}} = -1$$

• First Derivative Information:

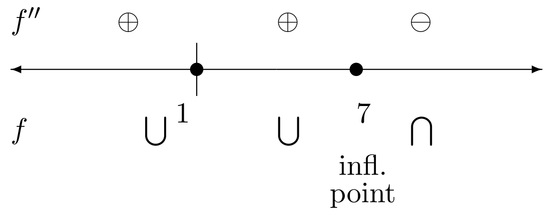
We know $f'(x) = \dfrac{x-5}{(x-1)^3}$. The critical points occur where f' is undefined or zero. The former happens when $x = 1$, but $x = 1$ was not in the domain of the original function, so it isn't technically a critical number. The latter happens when $x = 5$. As a result, $x = 5$ is the critical number. Using sign testing/analysis for f',



So f is decreasing on $(1, 5)$ and increasing on $(-\infty, 1)$ and $(5, \infty)$. Moreover, f has a local mi n at $x = 5$ with $f(5) = -\dfrac{9}{8}$.

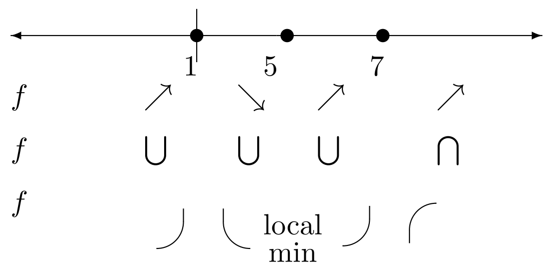
• Second Derivative Information:

Meanwhile, $f'' = \dfrac{-2x+14}{(x-1)^4}$. $f'' = 0$ when $x = 7$. Using sign testing/analysis for f'',

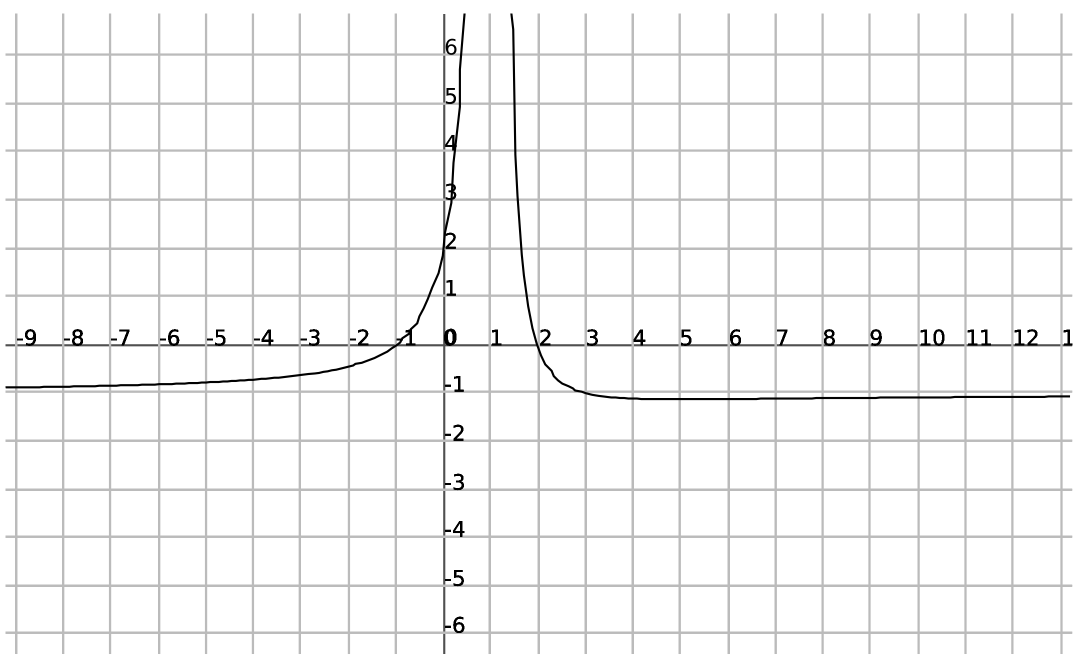


So f is concave down on $(7, \infty))$ and concave up on $(-\infty, 1)$ and $(1, 7)$. There is an inflection point at $(7, -\frac{10}{9})$.

• Piece the first and second derivative information together:



• Sketch:



6. [10 Points] **Position, Velocity, Acceleration**

A man stands on the edge of a bridge over a river. He throws a stone straight upward in the air with an initial velocity of 64 feet per second. The ball reaches a height of $\mathbf{s(t) = -16t^2 + 64t + 80}$ feet in t seconds above the water. Answer the following questions:

(a) What is the intitial height of the stone?

Initial position is $s(0) = 80$ feet.

(b) What is the maximum height that the stone reaches?

Max height occurs when $v(t) = 0$.

Compute $v(t) = -32t + 64 \overset{\text{set}}{=} 0$ or when $t = 2$ seconds.

Max height is $s(2) = -64 + 128 + 80 = 144$ feet above the water.

(c) What is the stone's velocity at time $t = 1$ second? Why is the velocity positive at time $t = 1$ second?

The stone's velicity at 1 second is given by $v(1) = -32 + 64 = 32$ ft/sec.

The velocity is postive because the object is moving up, that is, moving in the direction of increasing position.

(d) What is the stone's velocity at time $t = 3$ seconds? Why is the velocity negative at time $t = 3$ seconds?

The stone's velicity at 3 seconds is given by $v(3) = -32(3) + 64 = -96 + 64 = -32$ ft/sec.

The velocity is negative because the object is moving down, that is, moving in the direction of decreasing position.

(e) At what time will the stone hit the water? (Hint: position $s(t) = 0$)

The stone hits the water when $s(t) = -16t^2 + 64t + 80 = 0$ which factors

$-16(t^2 - 4t - 5) = -16(t-5)(t+1) = 0$ or when $t = 5$ or $t = -1$ (ignore negative time here).

So the stone hits the water when 5 seconds has passed.

(f) What is the stone's velocity when it hits the water?

If the stone hits the water at 5 seconds, then the velocity at impact is

$v(5) = -32(5) + 64 = -160 + 64 = -96$ feet per second.

(g) What is the stone's acceleration at any time t? The stone's acceleration at any time t is given by $a(t) = -32$ ft/sec^2.