HOMEWORK #16 ANSWER KEY

Math 105 Review Packet for Exam #2

Due Wednesday October 30 at the beginning of class.

Derivatives: Compute the derivative of each of the following functions **two** different ways:

- First use the **limit definition of the derivative**.
- Second use the Differentiation Rules.

1. $f(x) = \dfrac{1}{x^2}$

   Using the limit definition of the derivative, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim_{h\to 0}\frac{\left(\frac{x^2-(x+h)^2}{(x+h)^2x^2}\right)}{h}$$

$$\lim_{h\to 0}\left(\frac{x^2-(x^2+2xh+h^2)}{(x+h)^2x^2}\right)\cdot\left(\frac{1}{h}\right) = \lim_{h\to 0}\frac{x^2-x^2-2xh-h^2}{h(x+h)^2x^2} = \lim_{h\to 0}\frac{-2xh-h^2}{h(x+h)^2x^2}$$

$$= \lim_{h\to 0}\frac{h(-2x-h)}{h(x+h)^2x^2} = \lim_{h\to 0}\frac{-2x-h}{(x+h)^2x^2} = \frac{-2x}{x^4} = \boxed{\frac{-2}{x^3}}$$

   Or using the Power Rule we get

$$f'(x) = -2x^{-3} = \boxed{\frac{-2}{x^3}}$$

2. $f(x) = \sqrt{3x-7}$ Match!

   Using the limit definition of the derivative, we get

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{3(x+h)-7}-\sqrt{3x-7}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{3(x+h)-7}-\sqrt{3x-7}}{h}\cdot\frac{\sqrt{3(x+h)-7}+\sqrt{3x-7}}{\sqrt{3(x+h)-7}+\sqrt{3x-7}}$$

$$= \lim_{h\to 0}\frac{(3(x+h)-7)-(3x-7)}{h(\sqrt{3(x+h)-7}+\sqrt{3x-7})} = \lim_{h\to 0}\frac{3x+3h-7-3x+7}{h(\sqrt{3(x+h)-7}+\sqrt{3x-7})}$$

$$= \lim_{h\to 0}\frac{3h}{h(\sqrt{3(x+h)-7}+\sqrt{3x-7})}$$

$$= \lim_{h\to 0}\frac{3}{\sqrt{3(x+h)-7}+\sqrt{3x-7}} = \boxed{\frac{3}{2\sqrt{3x-7}}}$$

   Or using the Chain Rule we get

$$f'(x) = \frac{1}{2\sqrt{3x-7}}\cdot(3) = \boxed{\frac{3}{2\sqrt{3x-7}}}$$ Match!

3. $f(x) = \dfrac{3x+2}{5-6x}$

Using the limit definition of the derivative, we get

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{3(x+h)+2}{5-6(x+h)} - \frac{3x+2}{5-6x}}{h} = \lim_{h\to 0} \frac{\frac{3x+3h+2}{5-6x-6h} - \frac{3x+2}{5-6x}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{(3x+3h+2)(5-6x)-(3x+2)(5-6x-6h)}{(5-6x-6h)(5-6x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{15x+15h+10-18x^2-18xh-12x-(15x-18x^2-18xh+10-12x-12h)}{(5-6x-6h)(5-6x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{15x+15h+10-18x^2-18xh-12x-15x+18x^2+18xh-10+12x+12h}{(5-6x-6h)(5-6x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{15h+12h}{(5-6x-6h)(5-6x)} \cdot \frac{1}{h} = \lim_{h\to 0} \frac{27h}{(5-6x-6h)(5-6x)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{27}{(5-6x-6h)(5-6x)} = \boxed{\frac{27}{(5-6x)^2}}$$

Or using the Quotient Rule we get

$$f'(x) = \frac{(5-6x)(3)-(3x+2)(-6)}{(5-6x)^2} = \frac{15-18x+18x+12}{(5-6x)^2} = \boxed{\frac{27}{(5-6x)^2}} \quad \text{Match!}$$

Horizontal Tangent Lines: Find **all** x-coordinates at which the graphs of the following functions have horizontal tangent lines. Please **simplify** your derivatives first by factoring out common factors.

First note that we have horizontal tangent lines when the derivative (slope) equals 0. So take the derivative, simplify it, and then set it equal to 0 and solve for x.

4. $f(x) = \dfrac{1}{(x^2-3x)^7} = \left(x^2-3x\right)^{-7}$

$$f'(x) = -7(x^2-3x)^{-8}(2x-3) = -\frac{7(2x-3)}{(x^2-3x)^8} \overset{\text{set}}{=} 0$$

Recall that a fraction is equal to 0 when the numerator is equal to 0.

All you need to do is clear the denominator by multiplying both sides by the denominator.

That implies that $7(2x-3)=0$ which implies that $(2x-3)=0$ or finally $\boxed{x = \frac{3}{2}}$

5. $f(x) = (4x-1)^3(2x+7)^5$

Using the Product Rule and Chain Rules, we see that

$$\begin{aligned} f'(x) &= (4x-1)^3 5(2x+7)^4(2) + (2x+7)^5 3(4x-1)^2(4) \\ &= (4x-1)^2(2x+7)^4(2)\left[5(4x-1)+6(2x+7)\right] \\ &= (4x-1)^2(2x+7)^4(2)\left[20x-5+12x+42\right] \\ &= (4x-1)^2(2x+7)^4(2)\left[32x+37\right] \overset{\text{set}}{=} 0 \end{aligned}$$

Here one of the factors must be equal to 0.

$4x-1=0$ or $2x+7=0$ or $32x+37=0$

That means

$\boxed{x = \frac{1}{4}}$ or $\boxed{x = -\frac{7}{2}}$ or $\boxed{x = -\frac{37}{32}}$

Differentiation Rules: Differentiate the following functions. Please do **not** simplify your derivatives here.

6. $H(x) = \left(1 - \dfrac{2}{x^2}\right)^5$

$H'(x) = \boxed{5\left(1 - \dfrac{2}{x^2}\right)^4 \cdot \left(4x^{-3}\right)}$

7. $g(t) = \dfrac{t^3 + \frac{1}{t}}{1+t^2}$

$g'(t) = \boxed{\dfrac{(1+t^2)\left(3t^2 - \frac{1}{t^2}\right) - \left(t^3 + \frac{1}{t}\right)\cdot(2t)}{(1+t^2)^2}}$

8. $p(x) = \dfrac{1}{(-2x+3)^5}$ $\qquad p(x) = (-2x+3)^{-5}$, so

$p'(x) = \boxed{-5(-2x+3)^{-6}\cdot(-2)}$

9. $r(x) = \dfrac{(2x+1)^3}{(3x+1)^4}$

$r'(x) = \boxed{\dfrac{(3x+1)^4 \cdot 3(2x+1)^2\cdot(2) - (2x+1)^3\cdot 4(3x+1)^3\cdot(3)}{(3x+1)^8}}$

10. $S(x) = \left(\dfrac{1+2x}{1+3x}\right)^4$

$S'(x) = \boxed{4\left(\dfrac{1+2x}{1+3x}\right)^3 \cdot \left(\dfrac{(1+3x)\cdot 2 - (1+2x)\cdot 3}{(1+3x)^2}\right)}$

11. $y = \left((x^2+3x)^4 + x\right)^{-\frac{5}{7}}$

$y' = \boxed{-\frac{5}{7}\left[(x^2+3x)^4 + x\right]^{-12/7}[4(x^2+3x)^3(2x+3)+1]}$

Tangent Lines:

12. Find an equation for the tangent line to the graph of $y = x^2 - 4x + 2$ when $x = 1$.

First compute the derivative

$f'(x) = 2x - 4.$

The specific slope is $f'(1) = -2$.

Therefore, using *point slope form*, the equation of the tangent line through the point $(1, f(1)) = (1, -1))$ with slope equal to -2 is given by

$y - (-1) = -2(x - 1)$

or $y + 1 = -2x + 2$

or $\boxed{y = -2x + 1}$.

13. Find an equation for the tangent line to the graph of $f(x) = \dfrac{1}{x-1}$ at the point $(0, -1)$.

First we compute the derivative

$f'(x) = -\dfrac{1}{(x-1)^2}$

The specific slope is $f'(0) = -1$. Therefore, using *point slope form*, the equation of the tangent line through the point $(0, -1)$ with slope equal to -1 is given by

$y - (-1) = -1(x - 0)$

or $y + 1 = -1x$

or $\boxed{y = -x - 1}$.

14. Find an equation for the tangent line to the graph of $f(x) = \sqrt{x}$ at $x = 4$

First we compute the derivative

$f'(x) = \dfrac{1}{2\sqrt{x}}$

The specific clope is $f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$. The point is $(4, f(4)) = (4, \sqrt{4}) = (4, 2)$. Therefore, using *point slope form*, the equation of the tangent line throught the point $(4, 2)$ with slope $\dfrac{1}{4}$ is given by

$y - 2 = \dfrac{1}{4}(x - 4)$

or $y - 2 = \dfrac{1}{4}x - 1$

or $\boxed{y = \dfrac{1}{4}x + 1}$.

15. Find an equation for the tangent line to the graph of $f(x) = \dfrac{1+x^2}{\sqrt{2x+1}}$ at $x = 0$

First compute the derivative $f'(x) = \dfrac{\sqrt{2x+1}\ (2x) - (1+x^2)\dfrac{1}{\sqrt{2x+1}}}{\left(\sqrt{2x+1}\right)^2}$

$= \dfrac{\sqrt{2x+1}\ (2x) - (1+x^2)\dfrac{1}{\sqrt{2x+1}}}{2x+1}.$

No need to simplify here since we are going to evaluate this derivative function at $x = 0$ anyhow.

The specific clope is $f'(0) = \dfrac{0 - (1)\dfrac{1}{\sqrt{1}}}{\left(\sqrt{1}\right)^2} = \dfrac{-1}{1} = -1.$

The point is $(0, f(0)) = (0, 1)$. Therefore, using *point slope form*, the equation of the tangent line throught the point $(0, 1)$ with slope -1 is given by

$y - 1 = -1(x - 0)$

or $\boxed{y = -x + 1}$.

16. At which point(s) of the graph of $f(x) = -x^3 + 13$ is the slope of the tangent line equal to -27? What's the picture representing this problem?

First compute the derivative $f'(x) = -3x^2$.

Note: Set $f'(x) = -27$ and solve $f'(x) = -3x^2 = -27 \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$ so the points are $(3, f(3)) = \boxed{(3, -14)}$ and $(-3, f(-3)) = \boxed{(-3, 40)}$.

More derivatives:

17. Let f and g be two differentiable functions, and suppose that their values and the values of their derivatives at $x = 1, 2, 3$ are given by the following table:

x	1	2	3
$f(x)$	3	2	5
$f'(x)$	-2	1	3
$g(x)$	3	1	4
$g'(x)$	-3	2	7

Let $h(x) = f \circ g(x)$ and $k(x) = f(x) \cdot g(f(x))$. Compute $h'(2)$ and $k'(1)$.

Compute $h'(x) = f'(g(x))g'(x)$.

Then $h'(2) = f'(g(2))g'(2) = f'(1) \cdot 2 = (-2) \cdot 2 = \boxed{-4}$.

Compute $k'(x) = f(x) \cdot g'(f(x)) \cdot f'(x) + g(f(x)) \cdot f'(x)$.

Then $k'(1) = f(1) \cdot g'(f(1)) \cdot f'(1) + g(f(1)) \cdot f'(1)$

$= 3 \cdot g'(3) \cdot (-2) + g(3) \cdot (-2) = 3 \cdot 7 \cdot (-2) + 4 \cdot (-2) = -42 - 8 = \boxed{-50}$.

18. Let f and g be two differentiable functions, and suppose that their values and the values of their derivatives at $x = 2, 3$ are given by the following table:

x	2	3
$f(x)$	4	0
$f'(x)$	1	-7
$g(x)$	3	-1
$g'(x)$	-5	4

Let $h(x) = f(x)g(x), k(x) = \dfrac{f(x)}{g(x)}$ and $W(x) = f \circ g(x)$. Compute $h'(2)$ and $k'(2)$ and $W'(2)$.

Compute $h'(x) = f(x)g'(x) + f'(x)g(x)$.
Then $h'(2) = f(2)g'(2) + f'(2)g(2) = 4 \cdot (-5) + 1 \cdot 3 = \boxed{-17}$.

Compute $k'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

Then $k'(2) = \dfrac{g(2)f'(2) - f(2)g'(2)}{(g(2))^2} = \dfrac{3(1) - 4(-5)}{9} = \boxed{\dfrac{23}{9}}$.

Compute $W'(x) = f'(g(x)) \cdot g'(x)$.
Then $W'(2) = f'(g(2)) \cdot g'(2) = f'(3) \cdot (-5) = (-7)(-5) = \boxed{35}$.

19. Let $f(x) = \sqrt{x+1} \cdot g(x)$ where $g(0) = -7$ and $g'(0) = 4$. Compute $f'(0)$.

Compute $f'(x) = \sqrt{x+1} \cdot g'(x) + g(x)\dfrac{1}{2\sqrt{x+1}}$.

Then $f'(0) = \sqrt{0+1} \cdot g'(0) + g(0)\dfrac{1}{2\sqrt{0+1}} = 1 \cdot 4 + (-7) \cdot \dfrac{1}{2} = \dfrac{8}{2} - \dfrac{7}{2} = \boxed{\dfrac{1}{2}}$.

20. Let $f(x) = \dfrac{\sqrt{x^2+1}}{g(x)}$ where $g(0) = -7$ and $g'(0) = 4$. Compute $f'(0)$.

Compute $f'(x) = \dfrac{g(x)\ \dfrac{1}{2\sqrt{x^2+1}}(2x) - \sqrt{x^2+1}\ g'(x)}{(g(x))^2}$.

Then $f'(0) = \dfrac{g(0)\dfrac{1}{2\sqrt{1}}(0) - \sqrt{1}\ g'(0)}{(g(0))^2} = \boxed{-\dfrac{4}{49}}$

Implicit Differentiation: Use implicit differentiation to find the derivative of y with respect to x, for each of the following equations.

21. $x^2 + x^2y^2 + y^3 = 3x + 7$

Implicitly differentiate both sides.

$\dfrac{d}{dx}\left(x^2 + x^2y^2 + y^3\right) = \dfrac{d}{dx}(3x + 7)$

$2x + x^2(2y)\dfrac{dy}{dx} + y^2(2x) + 3y^2\dfrac{dy}{dx} = 3$

Isolate the $\dfrac{dy}{dx}$ terms.

$2x^2y\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 3 - 2x - 2xy^2$

Factor out the $\dfrac{dy}{dx}$ term.

$\left(2x^2y + 3y^2\right)\dfrac{dy}{dx} = 3 - 2x - 2xy^2$

Finally solve for $\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \boxed{\dfrac{3 - 2x - 2xy^2}{2x^2y + 3y^2}}$

22. $x^3 + x^2y^{\frac{3}{2}} = y^3 + 7$

Implicitly differentiate both sides.

$\dfrac{d}{dx}\left(x^3 + x^2y^{\frac{3}{2}}\right) = \dfrac{d}{dx}\left(y^3 + 7\right)$

$3x^2 + x^2\left(\dfrac{3}{2}\right)y^{\frac{1}{2}}\dfrac{dy}{dx} + y^{\frac{3}{2}}(2x) = 3y^2\dfrac{dy}{dx}$

Isolate the $\dfrac{dy}{dx}$ terms.

$\left(\dfrac{3}{2}\right)x^2\sqrt{y}\dfrac{dy}{dx} - 3y^2\dfrac{dy}{dx} = -3x^2 - 2xy^{\frac{3}{2}}$

Factor out the $\dfrac{dy}{dx}$ term.

$\left(\dfrac{3}{2}x^2\sqrt{y} - 3y^2\right)\dfrac{dy}{dx} = -3x^2 - 2xy^{\frac{3}{2}}$

Finally solve for $\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \boxed{\dfrac{-3x^2 - 2xy^{\frac{3}{2}}}{\frac{3}{2}x^2\sqrt{y} - 3y^2}}$