Math 105 **Answer Key** **Exam #2** **November 1, 2013**

1. [10 Points] Suppose that f and g are functions, **and**

- $\lim_{x\to 3} f(x) = 9$
- $g(7) = -6$
- $\lim_{x\to 4} f(x) = 7$
- $g(x)$ is continuous at $x = 7$
- $f(3) = -5$
- $f(x)$ is continuous at $x = 4$

Evaluate the following quantities and fully **justify** your answers. Do not just put down a value:

(a) $f(4) = \lim_{x\to 4} f(x) = \boxed{7}$ The first equality holds because of the assumption of f being continuous at $x = 4$. The second equality was given in the assumptions.

(b) $\lim_{x\to 7} g(x) = g(7) = \boxed{-6}$ The first equality holds because of the assumption of g being continuous at $x = 7$. The second equality was given in the assumptions.

(c) Compute $g \circ f(4) = g(f(4)) = g(7) = \boxed{-6}$ using the answer from (a) and the given values.

(d) Is $f(x)$ continuous at $x = 3$? Why or why not? Use math notation.

$\boxed{\text{No}}$, f is not continuous at $x = 3$ $\boxed{\text{because}}$ $9 = \lim_{x\to 3} f(x) \neq f(3) = -5$.

2. [35 Points] Compute the derivative of each of the following functions. For these problems, you do **NOT** need to simplify your derivative.

(a) $y = \frac{5}{6}x + x^{\frac{5}{6}} + \sqrt{5x+6} + \frac{1}{\sqrt{5x+6}} = \frac{5}{6}x + x^{\frac{5}{6}} + (5x+6)^{\frac{1}{2}} + (5x+6)^{-\frac{1}{2}}$.

$$y' = \boxed{\frac{5}{6} + \frac{5}{6}x^{-\frac{1}{6}} + \frac{5}{2\sqrt{5x+6}} + \left(-\frac{1}{2}\right)(5x+6)^{-\frac{3}{2}}(5)}$$

(b) $y = \left(\frac{x}{3} + \frac{5}{x^8}\right)^9$

$$y' = \boxed{9\left(\frac{x}{3} + \frac{5}{x^8}\right)^8 \left[\frac{1}{3} - \frac{40}{x^9}\right]}$$

(c) $f(x) = \left(x^2 - \frac{5}{x^2}\right)(3x + \sqrt{x})$

$$f'(x) = \boxed{\left(x^2 - \frac{5}{x^2}\right)\left(3 + \frac{1}{2\sqrt{x}}\right) + (3x + \sqrt{x})\left(2x + \frac{10}{x^3}\right)}$$

(d) $y = \frac{1}{\sqrt{x^2 - 5x + 3}} = \left(x^2 - 5x + 3\right)^{-\frac{1}{2}}$

$$y' = \boxed{-\frac{1}{2}\left(x^2-5x+3\right)^{-\frac{3}{2}}(2x-5)}$$

(e) $y = \left(\frac{1}{x^3}+7x\right)^{\frac{5}{7}}\left(x^4-\frac{1}{x^7}\right)^{-5}$

$$y' = \boxed{\left(\frac{1}{x^3}+7x\right)^{\frac{5}{7}}(-5)\left(x^4-\frac{1}{x^7}\right)^{-6}\left(4x^3+7x^{-8}\right)} \text{ continued...}$$

$$\boxed{+\left(x^4-\frac{1}{x^7}\right)^{-5}\left(\frac{5}{7}\right)\left(\frac{1}{x^3}+7x\right)^{-\frac{2}{7}}\left(-\frac{3}{x^4}+7\right)}$$

(f) $f(x) = \left(\dfrac{2\sqrt{x}+x^3}{x^{\frac{2}{3}}+\frac{2}{3}x}\right)^{\frac{2}{3}}$

$$f'(x) = \boxed{\frac{2}{3}\left(\frac{2\sqrt{x}+x^3}{x^{\frac{2}{3}}+\frac{2}{3}x}\right)^{-\frac{1}{3}}\left[\frac{\left(x^{\frac{2}{3}}+\frac{2}{3}x\right)\left(\frac{2}{2\sqrt{x}}+3x^2\right)-\left(2\sqrt{x}+x^3\right)\left(\frac{2}{3}x^{-\frac{1}{3}}+\frac{2}{3}\right)}{\left(x^{\frac{2}{3}}+\frac{2}{3}x\right)^2}\right]}$$

3. [10 Points] Find the equation of the tangent line to this curve $y = \sqrt{x+(x^2+1)^3}$ at the point where $x = 1$.

First we need slope. Compute the derivative.

$$y' = \frac{1}{2\sqrt{x+(x^2+1)^3}}\left(1+3(x^2+1)^2(2x)\right)$$

The specific slope at $x = 1$ is $y'(1) = \dfrac{1}{2\sqrt{1+(2)^3}}\left(1+3(2)^2(2)\right) = \dfrac{1}{2\sqrt{9}}(25) = \dfrac{25}{6}$

The point is $(1, y(1)) = (1, \sqrt{1+2^3}) = (1, \sqrt{9}) = (1, 3)$

Using point-slope form, we find the equation of the tangent line

$y - 3 = \frac{25}{6}(x-1)$ or $y - 3 = \frac{25}{6}x - \frac{25}{6}$

or finally

$$y = \frac{25}{6}x - \frac{25}{6} + 3$$

$$y = \frac{25}{6}x - \frac{25}{6} + \frac{18}{6}$$

$$\boxed{y = \frac{25}{6}x - \frac{7}{6}}$$

4. [15 Points] Consider the function $f(x) = \dfrac{7x+3}{1-5x}$.

(a) Compute the derivative of f using the **limit definition of the derivative**.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{7(x+h)+3}{1-5(x+h)} - \dfrac{7x+3}{1-5x}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{[7x+7h+3](1-5x) - (7x+3)[1-5x-5h]}{(1-5(x+h))(1-5x)}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{7x+7h+3-35x^2-35xh-15x-7x+35x^2+35xh-3+15x+15h}{(1-5(x+h))(1-5x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{22h}{(1-5(x+h))(1-5x)}\right)}{h} = \lim_{h\to 0} \frac{22h}{(1-5(x+h))(1-5x)} \left(\frac{1}{h}\right)$$

$$= \lim_{h\to 0} \frac{22}{(1-5(x+h))(1-5x)} = \boxed{\frac{22}{(1-5x)^2}}$$

(b) Compute the derivative of f using the Quotient Rule.

$$f'(x) = \frac{(1-5x)(7) - (7x+3)(-5)}{(1-5x)^2} = \frac{7-35x+35x+15}{(1-5x)^2} = \boxed{\frac{22}{(1-5x)^2}}$$

(c) Compute the second derivative $f''(x)$.

$f'(x) = 22(1-5x)^{-2}$

$$f''(x) = -44(1-5x)^{-3}(-5) = \boxed{\frac{220}{(1-5x)^3}}$$

5. [10 Points] Find **all** x-coordinates at which the graph of the function

$$f(x) = (4x+1)^4(7-3x)^8$$

has horizontal tangent lines.

$$\begin{aligned} f'(x) &= (4x+1)^4 8(7-3x)^7(-3) + (7-3x)^8 4(4x+1)^3(4) \\ &= 4(4x+1)^3(7-3x)^7\left[(-6)(4x+1) + 4(7-3x)\right] \\ &= 4(4x+1)^3(7-3x)^7\left[-24x-6+28-12x\right] \\ &= 4(4x+1)^3(7-3x)^7\left[-36x+22\right] \overset{\text{set}}{=} 0 \end{aligned}$$

This means that $4x+1=0$ or $7-3x=0$ or $-36x+22=0$

which implies $\boxed{x = -\frac{1}{4}}$ or $\boxed{x = \frac{7}{3}}$ or $\boxed{x = \frac{11}{18}}$

6. [10 Points] Find the equation of the line tangent to the curve $x^3 + x^2y = 6 - 4y^2$ at the point $(1, 1)$.

First, implicitly differentiate by taking the derivative of both sides:

$$\frac{d}{dx}\left(x^3 + x^2y\right) = \frac{d}{dx}\left(6 - 4y^2\right)$$

$$3x^2 + x^2\frac{dy}{dx} + y(2x) = 0 - 8y\frac{dy}{dx}$$

We can plug the point $(1, 1)$ immediately and solve for the derivative value $\frac{dy}{dx}$

$$3(1)^2 + (1)^2\frac{dy}{dx} + (1)(2(1)) = 0 - 8(1)\frac{dy}{dx}$$

$$3 + \frac{dy}{dx} + 2 = -8\frac{dy}{dx}$$

Isolate and solve for $\frac{dy}{dx}$

$$9\frac{dy}{dx} = -5$$

Finally, $\frac{dy}{dx} = -\frac{5}{9}$.

Now the equation of the tangent line at the point $(1, 1)$ with slope $-\frac{5}{9}$ is given by

$$y - 1 = -\frac{5}{9}(x - 1)$$

or $y - 1 = -\frac{5}{9}x - \frac{5}{9}$

or $y = -\frac{5}{9}x - \frac{5}{9} + 1$

or $y = -\frac{5}{9}x + \frac{5}{9} + \frac{9}{9}$

or $y = \boxed{-\frac{5}{9}x + \frac{14}{9}}$

7. [10 Points] For each of the following functions, compute the derivative of f **and simplify** your derivative as much as possible **into a single fraction**.

(a) $f(x) = \dfrac{x}{\sqrt{x^2+1}}$.

$$f'(x) = \frac{\sqrt{x^2+1}\,(1) - x\left(\dfrac{1}{2\sqrt{x^2+1}}(2x)\right)}{x^2+1} = \frac{\sqrt{x^2+1} - \left(\dfrac{x^2}{\sqrt{x^2+1}}\right)}{x^2+1}$$

$$= \frac{\left(\dfrac{\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)\sqrt{x^2+1} - \left(\dfrac{x^2}{\sqrt{x^2+1}}\right)}{x^2+1} \qquad \text{common denominator}$$

$$= \frac{\dfrac{x^2+1-x^2}{\sqrt{x^2+1}}}{x^2+1} = \frac{\dfrac{1}{\sqrt{x^2+1}}}{x^2+1} = \frac{1}{\sqrt{x^2+1}}\left(\frac{1}{x^2+1}\right) = \boxed{\frac{1}{(x^2+1)^{\frac{3}{2}}}}$$

(b) $f(x) = \left(\dfrac{x^2+1}{x^2-1}\right)^3$

$$f'(x) = 3\left(\frac{x^2+1}{x^2-1}\right)^2\left(\frac{(x^2-1)(2x)-(x^2+1)(2x)}{(x^2-1)^2}\right)$$

$$= 3\left(\frac{x^2+1}{x^2-1}\right)^2\left(\frac{2x^3-2x-2x^3-2x}{(x^2-1)^2}\right) = 3\left(\frac{(x^2+1)^2}{(x^2-1)^2}\right)\left(\frac{-4x}{(x^2-1)^2}\right)$$

$$= \boxed{\frac{-12x(x^2+1)^2}{(x^2-1)^4}}$$

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OPTIONAL BONUS

Do not attempt this unless you are completely done with the rest of the exam.

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OPTIONAL BONUS #1 Compute $\displaystyle\lim_{x\to 0} \frac{|x-1|-|x+1|-|x|}{|x|+|2-x|-|x+2|}$

$\boxed{\text{DOES NOT EXIST, RHL} \neq \text{LHL}}$

RHL: $\displaystyle\lim_{x\to 0} \frac{|x-1|-|x+1|-|x|}{|x|+|2-x|-|x+2|} = \lim_{x\to 0^+} \frac{-(x-1)-(x+1)-x}{x+(2-x)-(x+2)}$

$\displaystyle= \lim_{x\to 0^+} \frac{-x+1-x-1-x}{x+2-x-x-2} = \lim_{x\to 0^+} \frac{-3x}{-x} = \boxed{3}$

LHL: $\displaystyle\lim_{x\to 0} \frac{|x-1|-|x+1|-|x|}{|x|+|2-x|-|x+2|} = \lim_{x\to 0^-} \frac{-(x-1)-(x+1)-(-x)}{-x+(2-x)-(x+2)} =$

$\displaystyle= \lim_{x\to 0^-} \frac{-x+1-x-1+x}{-x+2-x-x-2} = \lim_{x\to 0^-} \frac{-x}{-3x} = \boxed{\frac{1}{3}}$

Here, recall that $|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$

$$|x-1| = \begin{cases} x-1 & \text{if} \quad x-1 \geq 0 \\ -(x-1) & \text{if} \quad x-1 < 0 \end{cases} = \begin{cases} x-1 & \text{if} \quad x \geq 1 \\ 1-x & \text{if} \quad x < 1 \end{cases}$$

$$|x+1| = \begin{cases} x+1 & \text{if} \quad x+1 \geq 0 \\ -(x+1) & \text{if} \quad x+1 < 0 \end{cases} = \begin{cases} x+1 & \text{if} \quad x \geq -1 \\ -x-1 & \text{if} \quad x < -1 \end{cases}$$

$$|2-x| = \begin{cases} 2-x & \text{if} \quad 2-x \geq 0 \\ -(2-x) & \text{if} \quad 2-x < 0 \end{cases} = \begin{cases} 2-x & \text{if} \quad x \leq 2 \\ x-2 & \text{if} \quad x > 2 \end{cases}$$

$$|x+2| = \begin{cases} x+2 & \text{if} \quad x+2 \geq 0 \\ -(x+2) & \text{if} \quad x+2 < 0 \end{cases} = \begin{cases} x+2 & \text{if} \quad x \geq -2 \\ -x-2 & \text{if} \quad x < -2 \end{cases}$$