Math 105 Practice Final Examination Fall, 2013

1. Evaluate each of the following limits. Please justify your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\displaystyle\lim_{x\to 1} \frac{x^2+x-6}{x^2-6x+8} \overset{\text{DSP}}{=} \boxed{-\frac{4}{3}}$

(b) $\displaystyle\lim_{x\to 6} \frac{x^2-4x-12}{|6-x|}$

$\boxed{\text{DNE}}$ since RHL$\neq$LHL. see below.

Note: $|6-x| = \begin{cases} 6-x & \text{if } x \le 6, \text{ that is } 6-x \ge 0 \\ -(6-x) & \text{if } x > 6, \text{ that is } 6-x < 0 \end{cases}$

RHL: $\displaystyle\lim_{x\to 6+} \frac{x^2-4x-12}{|6-x|} = \lim_{x\to 6+} \frac{x^2-4x-12}{-(6-x)} = \lim_{x\to 6+} \frac{(x-6)(x+2)}{x-6} = \lim_{x\to 6+} x+2 = 8$

LHL: $\displaystyle\lim_{x\to 6-} \frac{x^2-4x-12}{|6-x|} = \lim_{x\to 6-} \frac{x^2-4x-12}{6-x} = \lim_{x\to 6-} \frac{(x-6)(x+2)}{6-x} = \lim_{x\to 6-} -(x+2) = -8$

(c) $\displaystyle\lim_{x\to 1^+} \frac{x^2+x-2}{x^2-2x+1} = \lim_{x\to 1^+} \frac{(x+2)(x-1)}{(x-1)(x-1)} = \lim_{x\to 1^+} \frac{x+2}{x-1} = \frac{3}{0^+} = \boxed{+\infty}$

(d) $\displaystyle\lim_{x\to -7} \frac{\frac{7}{x}-\frac{1}{x+6}}{x+7} = \lim_{x\to -7} \frac{\left(\frac{7(x+6)-x}{x(x+6)}\right)}{x+7} = \lim_{x\to -7} \frac{7(x+6)-x}{x(x+6)(x+7)}$

$\displaystyle= \lim_{x\to -7} \frac{7x+42-x}{x(x+6)(x+7)} = \lim_{x\to -7} \frac{6x+42}{x(x+6)(x+7)} = \lim_{x\to -7} \frac{6(x+7)}{x(x+6)(x+7)}$

$\displaystyle= \lim_{x\to -7} \frac{6}{x(x+6)} = \frac{6}{-7(-1)} = \boxed{\frac{6}{7}}$

(e) $\displaystyle\lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+g(x+1)}$, where $g(x) = x^2+7$.

$\displaystyle\lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+g(x+1)} = \lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+(x+1)^2+7}$

$\displaystyle= \lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+(x^2+2x+1)+7} = \lim_{x\to 3^-} \frac{x^2-8x+15}{1-8x+x^2+2x+8}$

$\displaystyle= \lim_{x\to 3^-} \frac{x^2-8x+15}{x^2-6x+9} = \lim_{x\to 3^-} \frac{(x-5)(x-3)}{(x-3)(x-3)} = \lim_{x\to 3^-} \frac{x-5}{x-3} = \frac{-2}{0^-} = \boxed{+\infty}$

2. Compute each of the following derivatives.

(a) $f'(1)$, where $f(x) = \dfrac{x^2+1}{x\sqrt{x}+2x+1} = \dfrac{x^2+1}{x^{\frac{3}{2}}+2x+1}$ Simplify.

$$f'(x) = \frac{(x\sqrt{x}+2x+1)(2x) - (x^2+1)\left(\frac{3}{2}x^{\frac{1}{2}}+2\right)}{(x\sqrt{x}+2x+1)^2}$$

$$f'(1) = \frac{(1+2+1)(2) - (1+1)\left(\frac{3}{2}+2\right)}{(1+2+1)^2} = \frac{8-2\left(\frac{7}{2}\right)}{4^2} = \frac{8-7}{16} = \boxed{\frac{1}{16}}$$

(b) $\dfrac{d}{dx}\left(\dfrac{\sqrt{x^7-\frac{8}{x^3}}}{x^{\frac{9}{8}}-\frac{1}{x^{\frac{8}{9}}}+\frac{8}{9}x}\right)$ Do **not** simplify.

$$= \boxed{\frac{\left(x^{\frac{9}{8}}-\frac{1}{x^{\frac{8}{9}}}+\frac{8}{9}x\right)\frac{1}{2\sqrt{x^7-\frac{8}{x^3}}}\left(7x^6+\frac{24}{x^4}\right) - \sqrt{x^7-\frac{8}{x^3}}\left(\frac{9}{8}x^{\frac{1}{8}}+\frac{8}{9}x^{-\frac{17}{9}}+\frac{8}{9}\right)}{\left(x^{\frac{9}{8}}-\frac{1}{x^{\frac{8}{9}}}+\frac{8}{9}x\right)^2}}$$

(c) $g'''(x)$, where $g(x) = \dfrac{x}{1-2x}$ Simplify.

First $g'(x) = \dfrac{(1-2x)(1)-x(-2)}{(1-2x)^2} = \dfrac{1-2x+2x}{(1-2x)^2} = \dfrac{1}{(1-2x)^2} = (1-2x)^{-2}$

Second $g''(x) = (-2)(1-2x)^{-3}(-2) = 4(1-2x)^{-3}$

Third, $g'''(x) = -12(1-2x)^{-4}(-2) = \boxed{\dfrac{24}{(1-2x)^4}}$

(d) $\dfrac{dy}{dx}$, if $xy^3 + 3x^{\frac{7}{4}} = x^2y + 7$ Simplify.

$$x3y^2\frac{dy}{dx} + y^3(1) + \frac{21}{4}x^{\frac{3}{4}} = x^2\frac{dy}{dx} + y(2x) + 0$$

$$3xy^2\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy - y^3 - \frac{21}{4}x^{\frac{3}{4}}$$

$$\frac{dy}{dx}\left(3xy^2 - x^2\right) = 2xy - y^3 - \frac{21}{4}x^{\frac{3}{4}}$$

Finally, $\dfrac{dy}{dx} = \boxed{\dfrac{2xy - y^3 - \frac{21}{4}x^{\frac{3}{4}}}{3xy^2 - x^2}}$

(e) $g'(x)$, where $g(x) = \left(\frac{3x}{5} - \frac{3}{5x}\right)^{-5}\left(5x - \frac{\sqrt{x}}{3}\right)^{\frac{3}{5}}$ Do **not** simplify.

$$g'(x) = \boxed{\left(\frac{3x}{5} - \frac{3}{5x}\right)^{-5}\left(\frac{3}{5}\left(5x - \frac{\sqrt{x}}{3}\right)^{-\frac{2}{5}}\right)\left(5 - \frac{1}{6\sqrt{x}}\right)} \text{ (continued...)}$$

$$\boxed{+\left(5x - \frac{\sqrt{x}}{3}\right)^{\frac{3}{5}}\left((-5)\left(\frac{3x}{5} - \frac{3}{5x}\right)^{-4}\right)\left(\frac{3}{5} + \frac{3}{5x^2}\right)}$$

(f) $f'(x)$, where $f(x) = x^{\frac{1}{3}} + \frac{1}{x^{\frac{1}{3}}} + \frac{1}{1 + x^{\frac{1}{3}}} + \frac{1}{(1+x)^{\frac{1}{3}}} + \frac{1}{\left(1 + x^{\frac{1}{3}}\right)^{\frac{1}{3}}}$ Do **not** simplify.

$$f'(x) = \boxed{\frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{3}x^{-\frac{4}{3}} - \left(1 + x^{\frac{1}{3}}\right)^{-2}\left(\frac{1}{3}x^{-\frac{2}{3}}\right) - \frac{1}{3}(1+x)^{-\frac{2}{3}} - \frac{1}{3}\left(1 + x^{\frac{1}{3}}\right)^{-\frac{2}{3}}\left(\frac{1}{3}x^{-\frac{2}{3}}\right)}$$

3. Let $f(x) = \sqrt{7x - 3}$. Calculate $f'(x)$, using the **limit definition** of the derivative.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{7(x+h) - 3} - \sqrt{7x - 3}}{h}$$

$$= \lim_{h\to 0}\frac{\sqrt{7(x+h) - 3} - \sqrt{7x - 3}}{h} \cdot \frac{\sqrt{7(x+h) - 3} + \sqrt{7x - 3}}{\sqrt{7(x+h) - 3} + \sqrt{7x - 3}} = \lim_{h\to 0}\frac{(7(x+h) - 3) - (7x - 3)}{h(\sqrt{7(x+h) - 3} + \sqrt{7x - 3})}$$

$$= \lim_{h\to 0}\frac{7x + 7h - 3 - 7x + 3}{h(\sqrt{7(x+h) - 3} + \sqrt{7x - 3})} = \lim_{h\to 0}\frac{7h}{h(\sqrt{7(x+h) - 3} + \sqrt{7x - 3})}$$

$$= \lim_{h\to 0}\frac{7}{\sqrt{7(x+h) - 3} + \sqrt{7x - 3}} = \boxed{\frac{7}{2\sqrt{7x - 3}}}$$

Check your answer using the Chain Rule.

$$f'(x) = \boxed{\frac{7}{2\sqrt{7x - 3}}}$$

4. Consider the equation $y^2 + xy - x^5 = 8 - 8x - x^2 - y$.

Find the equation of the tangent line to this curve at the point where $(1, 0)$.

Implicitly differentiating:

$$2y\frac{dy}{dx} + x\frac{dy}{dx} + y(1) - 5x^4 = 0 - 8 - 2x - \frac{dy}{dx}$$

No need to solve explicitly for $\frac{dy}{dx}$ immediately. Just plug in the point $(1, 0)$.

$$0 + \frac{dy}{dx} + 0 - 5 = -8 - 2 - \frac{dy}{dx}$$

Then $2\frac{dy}{dx} = -5$.

Solve for slope: $\dfrac{dy}{dx} = -\dfrac{5}{2}$

Therefore, the equation of the tangent line through the point $(1, 0)$ with slope $-\dfrac{5}{2}$,

is given by $\quad y - 0 = -\dfrac{5}{2}(x-1)$ or $\boxed{y = -\dfrac{5}{2}x + \dfrac{5}{2}}$.

5. Find the absolute maximum and absolute minimum values of

$$f(x) = x^2\sqrt{5-x} \quad \text{on} \quad [1,5].$$

First compute the derivative

$$f'(x) = x^2\frac{1}{2\sqrt{5-x}}(-1) + \sqrt{5-x}(2x) = \frac{-x^2}{2\sqrt{5-x}} + \sqrt{5-x}(2x)\left(\frac{2\sqrt{5-x}}{2\sqrt{5-x}}\right)$$
$$= \frac{-x^2}{2\sqrt{5-x}} + \left(\frac{4x(5-x)}{2\sqrt{5-x}}\right) = \frac{-x^2+20x-4x^2}{2\sqrt{5-x}} = \frac{20x-5x^2}{2\sqrt{5-x}} = \frac{5x(4-x)}{2\sqrt{5-x}}.$$

$f'(x) \overset{\text{set}}{=} 0$ when $x = 0$ or $x = 4$.

$f'(x)$ is undefined at $x = 5$, which **is** in the domain of the original function, and just by chance one of the given closed interval endpoints.

So the critical numbers are $x = 0$, $x = 4$ and 5. Here $x = 0$ is not in the interval of interest.

Applying the Closed Interval method:

$f(4) = \boxed{16} \longleftarrow$ Absolute Maximum Value

$f(1) = 2$

$f(5) = \boxed{0} \longleftarrow$ Absolute Minimum Value

So the absolute maximum value is 16 (attained at $x = 4$), and the absolute minimum value is 0 (attained at $x = 5$).

6. Let $\boldsymbol{f(x) = \dfrac{x-3}{x^4}}$.

Sketch the graph of $y = f(x)$. State the domain for $f(x)$. Clearly indicate horizontal and vertical asymptotes, local minima/maxima, and inflection points on the graph, as well as where the graph is increasing, decreasing, concave up and concave down. Take my word that

$$\boldsymbol{f'(x) = \frac{3(4-x)}{x^5}} \quad \text{and} \quad \boldsymbol{f''(x) = \frac{12(x-5)}{x^6}}.$$

- Domain: $f(x)$ has domain $\{x | x \neq 0\}$
- VA: Vertical asymptotes $x = 0$. Don't need the following but, note that,

$\displaystyle\lim_{x\to 0^-} \frac{x-3}{x^4} = \frac{-3}{0^+} = -\infty$ and $\displaystyle\lim_{x\to 0^+} \frac{x-3}{x^4} = \frac{-3}{0^+} = -\infty$

- HA: Horizontal asymptote is $y = 0$ for this f since

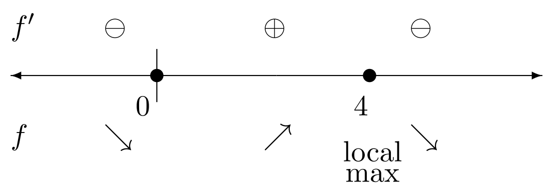
$$\lim_{x\to\pm\infty} \frac{x-3}{x^4} = \lim_{x\to\pm\infty} \frac{x}{x^4} - \frac{3}{x^4} = \lim_{x\to\pm\infty} \frac{1}{x^3} - \frac{3}{x^4} = 0 - 0 = 0$$

or because

$$\lim_{x\to\pm\infty} \frac{x-3}{x^4} \cdot \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} = \lim_{x\to\pm\infty} \frac{\frac{1}{x^3} - \frac{3}{x^4}}{1} = \frac{0}{1} = 0$$

• First Derivative Information:

We know $f'(x) = \dfrac{3(4-x)}{x^5}$. Set it equal to 0 and solve for x to find critical numbers. The critical points occur where f' is undefined or zero. The former happens when $x = 0$, but $x = 0$ was not in the domain of the original function, so it isn't technically a critical number. (We will still sign test around it.) The latter happens when $x = 4$. As a result, $x = 4$ is the critical number. Using sign testing/analysis for f',

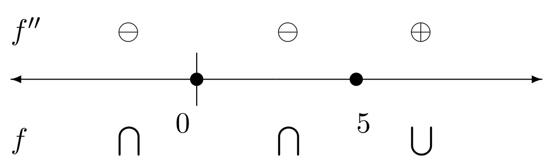


So f is decreasing on $(-\infty, 0)$ and $(4, \infty)$ and increasing on $(0, 4)$. Moreover, f has a local max at $(4, f(4)) = \left(4, \dfrac{1}{256}\right)$.

• Second Derivative Information:

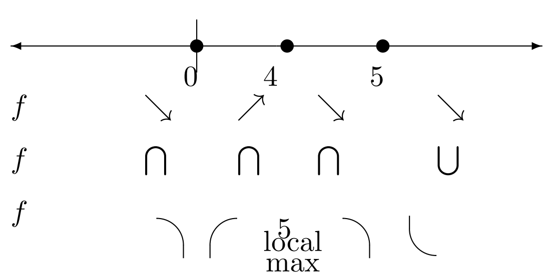
Meanwhile, $f'' = \dfrac{12(x-5)}{x^6}$.

We have a possible inflection point at $x = 5$. We note again that $x = 0$ is interesting here since it's not in the domain for f. (We will still sign test around it.) Using sign testing/analysis for f'',

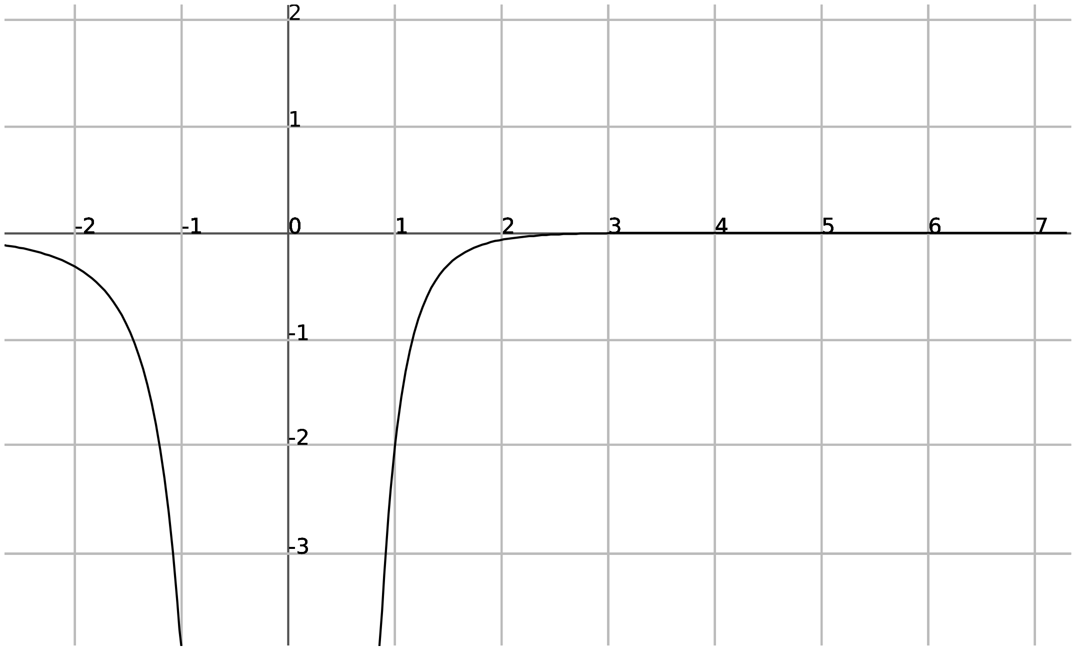


So f is concave down on $(-\infty, 0)$ and $(0, 5)$, and concave up on $(5, \infty)$ with an I.P at $(5, \frac{2}{5^4}) = (5, \frac{2}{625})$.

• Piece the first and second derivative information together:

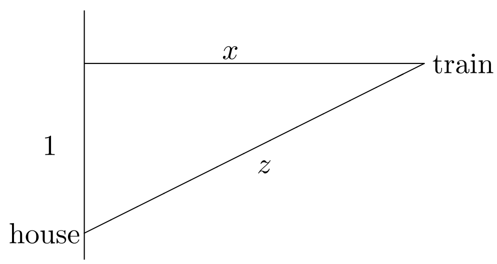


• Sketch:



7. A train is travelling east on a straight track at 40 mph. The track is crossed by a road going north and south, and a house is on the road one mile south of the track. Draw the straight line connecting the house to the train. How fast is this distance between the train and the house increasing when the train is 3 miles east of the road?

• Diagram



The picture at arbitrary time t is:

• Variables

Let $x =$ distance train has travelled horizontally(east) past the road at time t

Let $z =$ distance between train and house at time t

Find $\dfrac{dz}{dt}$ =? when $x = 3$ feet

and $\dfrac{dx}{dt} = 40\dfrac{\text{mi}}{\text{hr}}$

• Equation relating the variables:

Pythagorean theorem gives $x^2 + 1 = z^2$.

• Differentiate both sides w.r.t. time t.

$$2x\frac{dx}{dt} = 2z\frac{dz}{dt} \Longrightarrow x\frac{dx}{dt} = z\frac{dz}{dt}$$

• Substitute Key Moment Information (now and not before now!!!):

At the key instant when $x = 3$, using the Pythagorean Thrm, then $z = \sqrt{x^2+1} = \sqrt{3^2+1} = \sqrt{10}$

Then

$$x\frac{dx}{dt} = z\frac{dz}{dt}$$

becomes

$$3(40) = \sqrt{10}\frac{dz}{dt}$$

• Solve for the desired quantity:

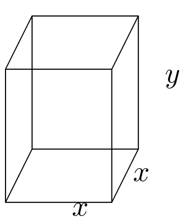
$$\frac{dz}{dt} = \frac{120}{\sqrt{10}} \text{ mph}$$

• Answer the question that was asked: The distance between the train and the house is growing at $\frac{120}{\sqrt{10}}$ miles every hour.

8. A large box with a square base and top is to be made to hold a fixed volume of 54 cubic feet. The sides cost \$1 per square foot. The top and bottom cost \$2 per square foot. Determine the dimensions that minimize the cost of materials.

(Remember to state the domain of the function you are computing extreme values for.)

• Diagram:



• Variables:

Let x =length of base of box.
Let y =height of the box.
Let V =volume of box.
Let C =cost of amount of material to make box.

• Equations:

We know the volume of the box given by $V = x^2y = 54$ is fixed, so that $y = \frac{54}{x^2}$.

Then the Cost of materials, which must be minimized, is given as

$$\begin{aligned} C &= \text{cost of bottom} + \text{cost of top} + \text{cost of 4 sides} \\ &= x^2(\$2) + x^2(\$2) + 4xy(\$1) \\ &= 4x^2 + 4xy \\ &= 4x^2 + 4x\left(\frac{54}{x^2}\right) \\ &= 4x^2 + \left(\frac{216}{x}\right) \end{aligned}$$

The (common-sense-bounds)domain of Cost is $\{x : x > 0\}$.

• Minimize:

Next $C' = 8x - \dfrac{216}{x^2}$. Setting $C' = 0$ we solve $x^3 = \dfrac{216}{8} = 27 \Longrightarrow x = 3$.

Sign-testing the critical number does indeed yield a minimum for the cost function.

C' ⊖ ⊕

C ↘ 3 ↗

MIN

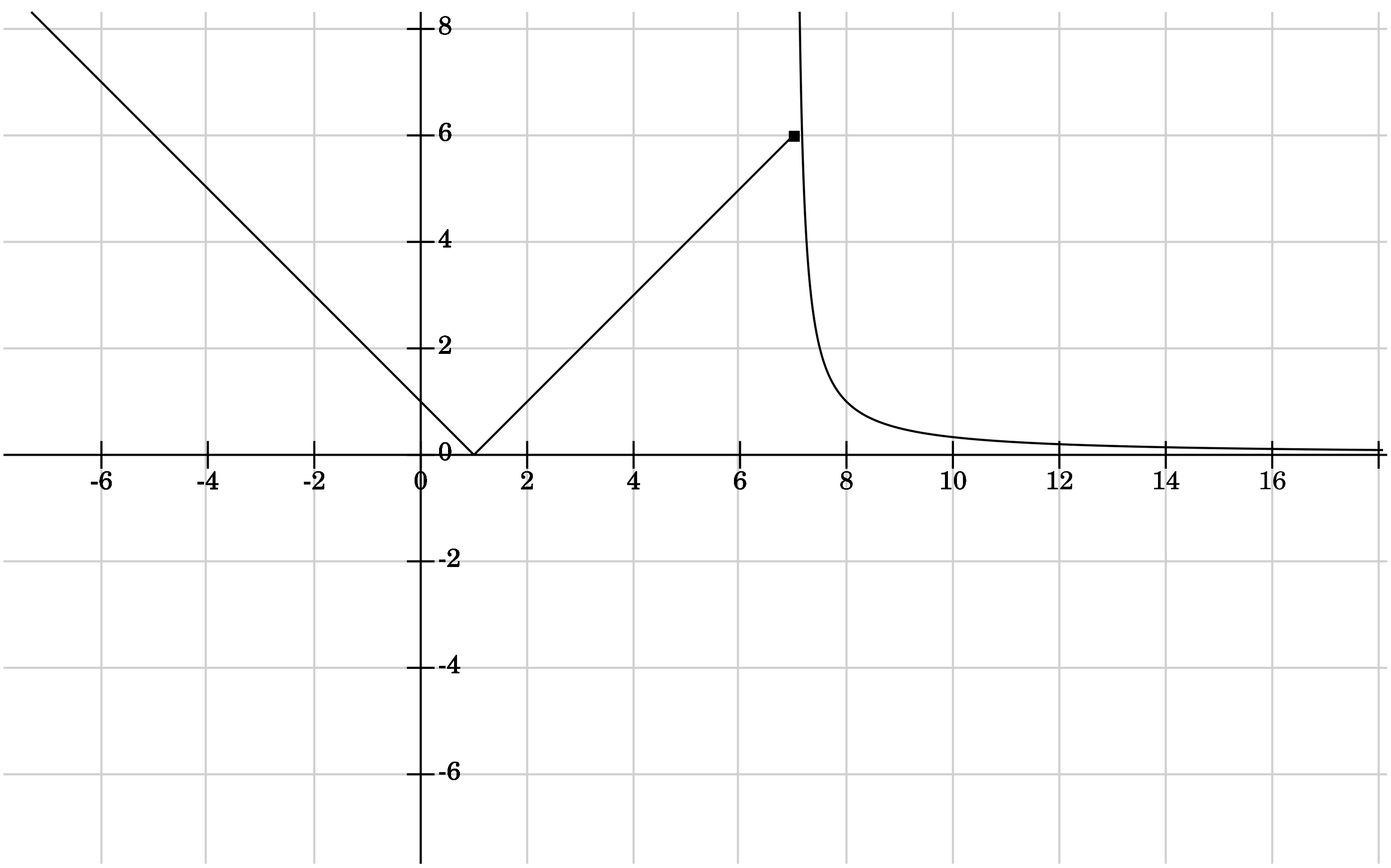
When $x = 3$ then $y = \dfrac{54}{(3)^2} = 6$.

• Answer: As a result, the most economical box has dimensions $\boxed{\text{3x3x6}}$ each in feet.

9. Consider the function defined by

$$f(x) = \begin{cases} |x-1| & \text{if } x \le 7 \\ \dfrac{1}{x-7} & \text{if } x > 7 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$.



(b) State the domain of the function $f(x)$.

Domain$=\{x|x \neq 7\}$

(c) Compute $\begin{cases} \lim_{x\to 7^+} f(x) = +\infty \\ \\ \lim_{x\to 7^-} f(x) = 6 \\ \\ \lim_{x\to 7} f(x) = \text{DNE since LHL} \neq \text{RHL} \end{cases}$

(d) State the value(s) of x at which f is discontinuous. Justify your answer(s) using the definition of continuity.

Note that $f(7)$ is defined, but $f(x)$ is discontinuous at $x = 7$ since $\lim_{x\to 7} f(x)$ DNE.

(e) State the value(s) of x where $f(x)$ is *not* differentiable. Justify your answer(s).

$f(x)$ is not differentiable at $x = 1$ because of the sharp corner there, but also not differentiable at $x = 7$ since it's not continuous there.