Worksheet 5 Answer Key

1. $$\int_0^{\pi/3} \sec^2\theta \, d\theta = \tan\theta \Big|_0^{\pi/3} = \cancel{\tan\frac{\pi}{3}}^{\sqrt{3}} - \cancel{\tan 0}^{0} = \sqrt{3} - 0 = \sqrt{3}$$

2. $$\int_{-\pi}^{\frac{\pi}{3}} 7\cos x \, dx = 7\int_{-\pi}^{\frac{\pi}{3}} \cos x \, dx = 7\sin x \Big|_{-\pi}^{\frac{\pi}{3}} = 7\left(\cancel{\sin\frac{\pi}{3}}^{\frac{\sqrt{3}}{2}} - \cancel{\sin(-\pi)}^{0}\right)$$
$$= 7\left(\frac{\sqrt{3}}{2} - 0\right) = \frac{7\sqrt{3}}{2}$$

3. $$\int_{-2}^{-1} x - \frac{5}{x^3} \, dx = \int_{-2}^{-1} x - 5x^{-3} \, dx = \frac{x^2}{2} \cancel{-} \frac{5x^{-2}}{\cancel{-}2} \Big|_{-2}^{-1} = \frac{x^2}{2} + \frac{5}{2x^2} \Big|_{-2}^{-1}$$
$$= \frac{\cancel{(-1)^2}^{1}}{2} + \frac{5}{2\cancel{(-1)^2}^{1}} - \left(\frac{\cancel{\cancel{(-2)^2}^{4}}^{2}}{\cancel{2}} + \frac{5}{2\cancel{(-2)^2}^{4}}\right)$$
$$= \underbrace{\frac{1}{2} + \frac{5}{2}}_{6/2 = 3} - \left(2 + \frac{5}{8}\right) = \underbrace{3 - 2}_{1} - \frac{5}{8} = \frac{8}{8} - \frac{5}{8} = \frac{3}{8}$$

4. $$\int_0^{\pi/6} (\tan x + \sec x) \sec x \, dx = \int_0^{\pi/6} \sec x \cdot \tan x + \sec^2 x \, dx = \sec x + \tan x \Big|_0^{\pi/6}$$
$$= \cancel{\sec\frac{\pi}{6}}^{\frac{2}{\sqrt{3}}} + \cancel{\tan\frac{\pi}{6}}^{\frac{1}{\sqrt{3}}} - \left(\cancel{\sec 0}^{1} + \cancel{\tan 0}^{0}\right)$$
$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} - 1 = \frac{3}{\sqrt{3}} - 1 = \sqrt{3} - 1$$

$$\sec\frac{\pi}{6} = \frac{1}{\cancel{\cos\frac{\pi}{6}}^{\frac{\sqrt{3}}{2}}} = \frac{2}{\sqrt{3}}$$

5. $$\int_1^2 \left(x-\frac{1}{x}\right)^2 dx = \int_1^2 \left(x-\frac{1}{x}\right)\left(x-\frac{1}{x}\right) dx = \int_1^2 x^2 - \cancel{\frac{x}{x}}^{-1} - \cancel{\frac{x}{x}}^{-1} + \frac{1}{x^2}\, dx$$

$$= \int_1^2 x^2 - 2 + x^{-2}\, dx = \frac{x^3}{3} - 2x + \frac{x^{-1}}{-1}\Bigg|_1^2 = \frac{x^3}{3} - 2x - \frac{1}{x}\Bigg|_1^2$$

$$= \frac{8}{3} - 4 - \frac{1}{2} - \left(\frac{1}{3} - 2 - 1\right) = \frac{8}{3} - 4 - \frac{1}{2} - \frac{1}{3} + 2 + 1$$

$$= \frac{7}{3} - \frac{1}{2} - 1 = \frac{14}{6} - \frac{3}{6} - \frac{6}{6} = \frac{5}{6}$$

6. $$\int_0^{16} \frac{1}{x^{3/4}} - \frac{2}{\sqrt{x}}\, dx = \int_0^{16} x^{-3/4} - 2x^{-1/2}\, dx = \frac{x^{1/4}}{1/4} - 2\frac{x^{1/2}}{1/2}\Bigg|_0^{16}$$

$$= 4x^{1/4} - 4\sqrt{x}\Bigg|_0^{16} = 4\cancel{(16)^{1/4}}^{2} - 4\cancel{\sqrt{16}}^{4} - (0-0)$$

$$= 8 - 16 = -8$$

7. $$\int_1^4 \frac{\sqrt{x} - x^2}{x}\, dx = \int_1^4 \frac{\sqrt{x}}{x} - \frac{x^2}{x}\, dx = \int_1^4 x^{-\frac{1}{2}} - x\, dx = \frac{x^{1/2}}{1/2} - \frac{x^2}{2}\Bigg|_1^4$$

split

$$= 2\sqrt{x} - \frac{x^2}{2}\Bigg|_1^4 = 2\cancel{\sqrt{4}}^{2} - \cancel{\frac{16}{2}}^{-8} - \left(2\cancel{\sqrt{1}} - \frac{1}{2}\right)$$

2

$$= 4 - 8 - 2 + \frac{1}{2} = -6 + \frac{1}{2} = \frac{-12}{2} + \frac{1}{2} = \frac{-11}{2}$$

8. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{3 + \cos^2 x}{\cos^2 x}\, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{3}{\cos^2 x} + \cancel{\frac{\cos^2 x}{\cos^2 x}}^{1}\, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 3\sec^2 x + 1\, dx$$

split

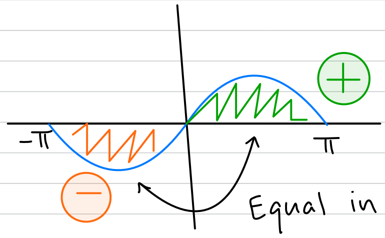
$$= 3\tan x + x\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \left(3\cancel{\tan\frac{\pi}{3}}^{\sqrt{3}} + \frac{\pi}{3}\right) - \left(3\cancel{\tan\frac{\pi}{4}}^{1} + \frac{\pi}{4}\right)$$

$$= 3\sqrt{3} + \underbrace{\frac{\pi}{3}}_{\frac{4\pi}{12}} - 3 - \underbrace{\frac{\pi}{4}}_{-\frac{3\pi}{12}} = 3\sqrt{3} - 3 + \frac{\pi}{12}$$

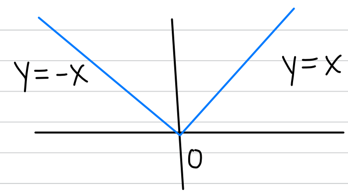
Match!

9. $\int_{-\pi}^{\pi} \sin x \, dx = -\cos x \Big|_{-\pi}^{\pi} = -\cos\pi + \cos(-\pi) = 1 - 1 = 0$ Match

Makes sense because the Definite Integral computes the Area bounded Above x-axis MINUS Area bounded Below x-axis



10(a) $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ Cases

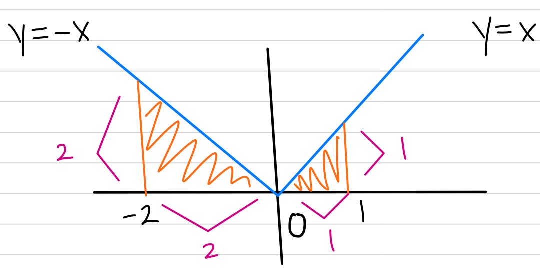


10(b) $\int_{-2}^{1} |x| \, dx = \int_{-2}^{0} \cancel{|x|}^{\,-x} dx + \int_{0}^{1} \cancel{|x|}^{\,x} dx$

$= \int_{-2}^{0} -x \, dx + \int_{0}^{1} x \, dx$ run each separately

$= -\frac{x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_{0}^{1}$

$= -0 \cancel{-}^{+}\left(\cancel{-}\frac{(-2)^2}{2}\right) + \frac{1}{2} - 0 = 2 + \frac{1}{2} = \frac{5}{2}$



Note: All bounded Area is above x-Axis ⇒ +

Area $= \frac{1}{2}\cdot 2\cdot 2 \qquad \frac{1}{2}\cdot 1\cdot 1$

$2 + \frac{1}{2} = \frac{5}{2}$ Match

FTC First

$$11(a)\ \int_2^5 x^2\,dx = \left.\frac{x^3}{3}\right|_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{125}{3} - \frac{8}{3} = \frac{117}{3} = 39$$

Limit Definition

$$11(b)\ \int_2^5 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(2+\frac{3i}{n}\right)\cdot\left(\frac{3}{n}\right)$$

Here $f(x) = x^2$

$a = 2 \quad b = 5$

$\Delta x = \frac{b-a}{n} = \frac{5-2}{n} = \frac{3}{n}$

$x_i = a + i\Delta x$

$= 2 + i\left(\frac{3}{n}\right)$

$= 2 + \frac{3i}{n}$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} f\left(2+\frac{3i}{n}\right)$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(2+\frac{3i}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 + \frac{12i}{n} + \frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 + \sum_{i=1}^{n} \frac{12i}{n} + \sum_{i=1}^{n} \frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 4 + \frac{3}{n} \sum_{i=1}^{n} \frac{12i}{n} + \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{12}{n} \sum_{i=1}^{n} 1 + \frac{36}{n^2} \sum_{i=1}^{n} i + \frac{27}{n^3} \sum_{i=1}^{n} i^2$$

$$= \lim_{n\to\infty} \frac{12}{\cancel{n}} \cdot \cancel{n} + \frac{36}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \lim_{n\to\infty} 12 + \frac{\overset{18}{\cancel{36}}}{\cancel{2}}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) + \frac{27}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} 12 + 18(1)\cdot\left(1+\frac{1}{n}\right) + \frac{27}{6}\cdot(1)\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)$$

$$= 12 + 18\cdot 1\cdot 1 + \frac{\overset{9}{\cancel{27}}}{\underset{3}{\cancel{6}}}\cdot 1\cdot 1\cdot \cancel{2}$$

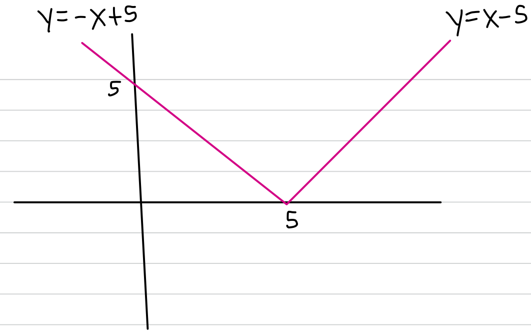
$$= 12 + 18 + 9 = 39$$

Match!

Cases

12(a) $f(x) = |x-5| = \begin{cases} x-5 & \text{if } x-5 \geq 0 \\ -(x-5) & \text{if } x-5 < 0 \end{cases}$

$$\begin{cases} x-5 & \text{if } x \geq 5 \\ -x+5 & \text{if } x < 5 \end{cases}$$



12(b) $$\int_4^7 |x-5|\,dx = \int_4^5 \overset{-x+5}{\cancel{|x-5|}}\,dx + \int_5^7 \overset{x-5}{\cancel{|x-5|}}\,dx$$

$$= \int_4^5 -x+5\,dx + \int_5^7 x-5\,dx$$

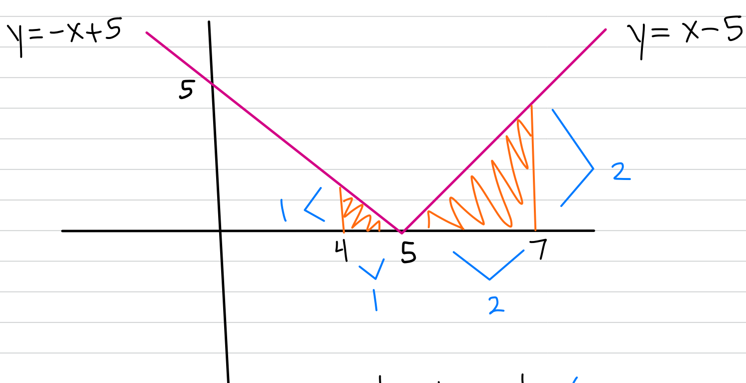
$$= -\frac{x^2}{2} + 5x \Big|_4^5 + \frac{x^2}{2} - 5x \Big|_5^7$$

$$= \left(-\frac{25}{2} + 25\right) - \left(-\frac{\cancel{16}}{2}^{-8} + 20\right) + \left(\frac{49}{2} - 35\right) - \left(\frac{25}{2} - 25\right)$$

$$= -\frac{25}{2} + \cancel{25} + 8 - 20 + \frac{49}{2} \underbrace{- 35 - \frac{25}{2} + 25}_{-10}$$

$$-\frac{50}{2} = -\cancel{25}$$

$$= -12 + \frac{49}{2} - 10 = -22 + \frac{49}{2} = -\frac{44}{2} + \frac{49}{2} = \frac{5}{2}$$



$$\text{Area} = \frac{1}{2}\cdot 1 \cdot 1 + \frac{1}{\cancel{2}}\cdot \cancel{2}\cdot 2$$

$$= \frac{1}{2} + 2 = \frac{5}{2}$$ Match