Math 106, Spring 2022

Homework #8

Due **Friday, March 11th** in Gradescope by 11:59 pm ET

Goal: Computing Area using the Limit Definition of the Definite Integral using Riemann Sum

Definition: the **Definite Integral** of a function f from $x = a$ to $x = b$ is given by

$$(\bullet) \quad \int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \lim_{n\to\infty} \left[f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_i)\Delta x + \ldots + f(x_n)\Delta x \right]$$

Note: The Definite Integral is a Limit of a Sum of areas! Just think about this formula as

the **Limiting Value** *of the sum of the areas of finitely many (n) approximating rectangles.*

To compute definite integrals the *long (limit) way*, follow these steps:

Step 1: Given the integral $\int_a^b f(x)\, dx$, **pick off** or **identify** the **integrand** $f(x)$, and **limits of integration** a and b.

Step 2: Compute $\boxed{\Delta x = \frac{b-a}{n}}$. This Width of each partitioned interval should be in terms of n.

Step 3: Compute $\boxed{x_i = a + i\Delta x}$. Leave the i as your counter. You have the left-most endpoint a from Step 1. You have width Δx from Step 2. This endpoint x_i should be in terms of i and n.

Step 4: Plug x_i and Δx into the formula $(\bullet)$ above. Here it is again:

$$(\bullet) \quad \boxed{\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x} \quad \longleftarrow \textbf{MEMORIZE!}$$

Step 5: Use the following formulas for sum of integers i and finish evaluating the limit in n.

$$\sum_{i=1}^{n} 1 = n \qquad (*) \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad (**) \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(***) \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Note: your final answer for the definite integral should be a **number** after you finish the limit.

Read through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i. Because it doesn't feel natural yet, just trust the formulas right now.

Evaluate $\displaystyle\int_0^6 x^2\,dx$ using the Limit Definition of the Definite Integral and Riemann Sums.

Here $f(x) = x^2$, $a = 0$, $b = 6$, $\Delta x = \dfrac{b-a}{n} = \dfrac{6-0}{n} = \dfrac{6}{n}$ and $x_i = a + i\Delta x = 0 + i\left(\dfrac{6}{n}\right) = \dfrac{6i}{n}$.

$$
\begin{aligned}
\int_0^6 x^2 dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \quad &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(\frac{6i}{n}\right)\frac{6}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\left(\frac{6i}{n}\right)^2\right)\frac{6}{n} \\
&= \lim_{n\to\infty}\frac{6}{n}\sum_{i=1}^{n}\frac{36i^2}{n^2} \\
&= \lim_{n\to\infty}\left(\frac{216}{n^3}\sum_{i=1}^{n} i^2\right) \\
&= \lim_{n\to\infty}\left(\frac{216}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \text{ using } (**) \\
&= \lim_{n\to\infty}\left(\frac{216}{6}\cdot\frac{n(n+1)(2n+1)}{n^3}\right) \\
&= \lim_{n\to\infty}\left(\frac{216}{6}\cdot\frac{n(n+1)(2n+1)}{n\cdot n\cdot n}\right) \\
&= \lim_{n\to\infty}\left(\frac{216}{6}\cdot\left(\frac{n}{n}\right)\cdot\left(\frac{n+1}{n}\right)\cdot\left(\frac{2n+1}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{216}{6}\cdot 1\cdot\left(1+\frac{1}{n}\right)\cdot\left(2+\frac{1}{n}\right)\right) \\
&= \frac{216}{6}\cdot 1\cdot 2 = \frac{216}{3} = \boxed{72}
\end{aligned}
$$

Read through the entire next problem. Make sure you understand the formula to start, as well as the formulas for Δx and x_i. Here the lower limit of integration a is **not** 0.

Evaluate $\displaystyle\int_1^4 6-3x\ dx$ using the Limit Definition of the Definite Intergal and Riemann Sums.

Here $f(x) = 6-3x,\ a=1,\ b=4,\ \Delta x = \dfrac{b-a}{n} = \dfrac{4-1}{n} = \dfrac{3}{n}$

and $x_i = a + i\Delta x = 1 + i\left(\dfrac{3}{n}\right) = 1 + \dfrac{3i}{n}$.

$$\begin{aligned}
\int_1^4 6-3x\ dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x \quad &= \lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{3i}{n}\right)\frac{3}{n} \\
&= \lim_{n\to\infty}\sum_{i=1}^{n}\left(6-3\left(1+\frac{3i}{n}\right)\right)\frac{3}{n} \\
&= \lim_{n\to\infty}\left(\frac{3}{n}\sum_{i=1}^{n}\left(3-\frac{9i}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{3}{n}\left(\sum_{i=1}^{n}3-\sum_{i=1}^{n}\frac{9i}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(\frac{9}{n}\sum_{i=1}^{n}1-\frac{27}{n^2}\sum_{i=1}^{n}i\right) \\
&= \lim_{n\to\infty}\left(\frac{9}{n}(n)-\frac{27}{n^2}\frac{n(n+1)}{2}\right) \text{ using } (*) \\
&= \lim_{n\to\infty}\left(9-\frac{27}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\right) \\
&= \lim_{n\to\infty}\left(9-\frac{27}{2}(1)\left(1+\frac{1}{n}\right)\right) \\
&= 9-\frac{27}{2} \\
&= \boxed{-\frac{9}{2}}
\end{aligned}$$

Now complete these Homework problems:

1. Compute by hand, manually, the Area bounded above by the graph of $y = 2x + 5$ and below by $y = 0$ and between $x = 0$ and $x = 3$. Sketch the graph and shade the bounded region.

2. Evaluate $\displaystyle\int_0^3 2x + 5\ dx$ using the Limit Definition of the Definite Integral and Riemann Sums

3. Compute by hand, manually, the **Net Area** bounded between the graph of $y = 4 - 2x$ and the x-axis $(y = 0)$ and between $x = 1$ and $x = 5$. Sketch the graph and shade the bounded region.

4. Evaluate $\displaystyle\int_1^5 4 - 2x\ dx$ using the Limit Definition of the Definite Integral and Riemann Sums

NOTE: Recall the Definite Integral computes the Area bounded above the x-axis *minus* the Area bounded below the x-axis.

5. Evaluate $\displaystyle\int_0^4 x^2\ dx$ using the Limit Definition of the Definite Integral and Riemann Sums. Sketch the graph and shade the bounded region.

6. Evaluate $\displaystyle\int_{-1}^2 x^2 - 3x + 2\ dx$ using the Limit Definition of the Definite Integral and Riemann Sums. Sketch the graph and shade the bounded region.

REGULAR OFFICE HOURS

Monday: 1:00–3:00 pm

Tuesday: 12:00–4:00 pm

7:30–9:000 pm TA Bobby, SMUDD 205

Wednesday: 1:00-3:00 pm

Thursday: none for Professor

7:30–9:000 pm TA Bobby, SMUDD 205

Friday: 12:00–2:00 pm

- Enjoy your Spring Vacation!!