Exam 3 Spring 2022

1. $$\frac{d}{dx}\left(\frac{\ln x\,\sqrt{1+e^{x}}}{(4-x^{6})^{3}\;e^{-\cos x}}\right)$$

use Log Algebra Rules

1st Simplify

$$= \frac{d}{dx}\;\ln\left(\ln x\cdot\sqrt{1+e^{x}}\right) - \ln\left((1-x^{6})^{3}\,e^{-\cos x}\right)$$

$-\cos x$

$$= \frac{d}{dx}\;\ln(\ln x) + \ln\left((1+e^{x})^{\frac{1}{2}}\right) - \left(\ln\left((1-x^{6})^{3}\right) + \cancel{\ln\left(e^{-\cos x}\right)}\right)$$

$$= \frac{d}{dx}\;\ln(\ln x) + \frac{1}{2}\ln(1+e^{x}) - 3\ln(1-x^{6}) + \cos x$$

2nd Differentiation ↳ Chain Rule

$$= \frac{1}{\ln x}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{1+e^{x}}\cdot e^{x} - 3\,\frac{1}{1-x^{6}}\cdot(-6x^{5}) - \sin x$$

$$= \boxed{\frac{1}{x\ln x} + \frac{e^{x}}{2(1+e^{x})} + \frac{18x^{5}}{1-x^{6}} - \sin x}$$

2. Logarithmic Differentiation

$y = x^{x}$ Take log of both sides

$$\ln y = \ln\left(x^{x}\right)$$

$$\ln y = x\cdot\ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x\cdot\ln x)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \cancel{x\cdot\frac{1}{x}} + \ln x\cdot(1)$$

$$\frac{dy}{dx} = \cancel{y}^{\,x^{x}}(1+\ln x) = \boxed{x^{x}(1+\ln x)}$$

3(a) $f(x) = \sqrt{\ln x} - \ln\sqrt{x}$

OR// 2nd piece

$\ln\sqrt{x} = \ln(x^{1/2}) = \frac{1}{2}\ln x$

$\hookrightarrow \frac{1}{2}\cdot\frac{1}{x}$ Match

$$f'(x) = \frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x} - \underbrace{\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}}_{2x}$$

$$f'(e^4) = \frac{1}{2\sqrt{\cancel{\ln(e^4)}_{4}}\cdot e^4} - \frac{1}{2e^4} = \frac{1}{4e^4} - \frac{2}{4e^4} = \frac{-1}{4e^4}$$

$\sqrt{4} = 2$

Match!

3(b) $f(x) = e^{2x} + \frac{1}{e^{2x}} = e^{2x} + e^{-2x}$

$$f'(x) = e^{2x}\cdot 2 - 2e^{-2x} = 2e^{2x} - \frac{2}{e^{2x}}$$

$$f'(\ln 3) = 2e^{2\ln 3} - \frac{2}{e^{2\ln 3}}$$

$$= 2\cancel{e}^{\ln(3^2)}{}_{9} - \frac{2}{\cancel{e}^{\ln(3^2)}}{}_{9}$$

$$= 18 - \frac{2}{9} = \frac{162}{9} - \frac{2}{9} = \frac{160}{9}$$

Match!

4. $f(x) = \sin(\ln(1+x)) - \ln(1+\sin(5x)) - e^{\cos x} - \sin(e^{3x}-1)$

$$f'(x) = \cos(\ln(x+1))\cdot\frac{1}{x+1} - \frac{1}{1+\sin(5x)}\cdot\cos(5x)\cdot 5 - e^{\cos x}\cdot(-\sin x) - \cos(e^{3x}-1)\cdot e^{3x}\cdot 3$$

$$f'(0) = \cos(\ln 1)\cdot\frac{1}{1} - \frac{1}{1+\sin 0}\cdot\cos 0\cdot 5 + e^{\cos 0}\cdot\sin 0 - \cos(e^0-1)\,e^0\cdot 3$$

$1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 5 \quad e \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 3$

$$= 1 - 5 + 0 - 3 = -7$$

Match!

5. $f(x) = e^{6x} + \frac{6}{e^{6x}} + \overbrace{\cancel{e^{\ln 6}}}^{\text{Constant}} - \frac{6}{x} + \frac{1}{e^{6x}} - \overbrace{\cancel{\ln(e^{6})}}^{\text{Constant}} + 6e^{6x} - \frac{e}{x^{6}} + \frac{e^{x}}{e^{6x}} + (e^{6x})(e^{x})$

Prep

$$= e^{6x} + 6e^{-6x} + 6 - 6x^{-1} + e^{-6x} - 6 + 6e^{6x} - e \cdot x^{-6} + e^{-5x} + e^{7x}$$

$$f'(x) = \boxed{6e^{6x} - 36e^{-6x} + 0 + 6x^{-2} - 6e^{-6x} + 0 + 36e^{6x} + 6e\,x^{-7} - 5e^{-5x} + 7e^{7x}}$$

6(a) $\displaystyle\int e^{2x}(2+e^{2x})^{7}\,dx = \frac{1}{2}\int u^{7}\,du = \frac{1}{2}\cdot\frac{u^{8}}{8} + C = \frac{u^{8}}{16} + C$

$$\boxed{\begin{aligned} u &= 2+e^{2x} \\ du &= 2e^{2x}\,dx \\ \tfrac{1}{2}\,du &= e^{2x}\,dx \end{aligned}}$$

$$= \boxed{\frac{(2+e^{2x})^{8}}{16} + C}$$

6(b) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^{u}\,du = 2e^{u} + C = \boxed{2e^{\sqrt{x}} + C}$

$$\boxed{\begin{aligned} u &= \sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \\ 2\,du &= \frac{1}{\sqrt{x}}\,dx \end{aligned}}$$

6(c) $\displaystyle\int \frac{(1+e^{2x})^{2}}{e^{6x}}\,dx = \int \frac{1+2e^{2x}+e^{4x}}{e^{6x}}\,dx = \int e^{-6x} + 2e^{-4x} + e^{-2x}\,dx$

split

FOIL Algebra
+
Split-Split

$$= \frac{e^{-6x}}{-6} + \frac{\cancel{2}e^{-4x}}{-\cancel{4}} + \frac{e^{-2x}}{-2} + C$$

using k-rule

$\int e^{kx}dx = \frac{e^{kx}}{k} + C$

OR $$= \boxed{-\frac{1}{6e^{6x}} - \frac{1}{2e^{4x}} - \frac{1}{2e^{2x}} + C}$$

7(a) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{e+\cos x}\,dx = -\int_{e}^{e-1} \frac{1}{u}\,du = -\ln|u|\Big|_{e}^{e-1}$

$$\boxed{\begin{aligned} u &= e+\cos x \\ du &= -\sin x\,dx \\ -du &= \sin x\,dx \end{aligned}}$$

$= -\ln|e-1| + \cancelto{1}{\ln|e|}$

$= \boxed{1-\ln|e-1|}$ Match!

$$\boxed{\begin{aligned} x=\tfrac{\pi}{2} &\Rightarrow u = e + \cancelto{0}{\cos\tfrac{\pi}{2}} = e \\ x=\pi &\Rightarrow u = e + \cancelto{-1}{\cos\pi} = e-1 \end{aligned}}$$

7(b) $\displaystyle\int_{0}^{\ln 3} \frac{e^{2x}}{1+e^{2x}}\,dx = \frac{1}{2}\int_{2}^{10} \frac{1}{u}\,du = \frac{1}{2}\ln|u|\Big|_{2}^{10} = \frac{1}{2}\left(\ln(10)-\ln 2\right)$

$$\boxed{\begin{aligned} u &= 1+e^{2x} \\ du &= 2e^{2x}\,dx \\ \tfrac{1}{2}\,du &= e^{2x}\,dx \end{aligned}}$$

$= \frac{1}{2}\ln\left(\frac{10}{2}\right)$

$= \frac{1}{2}\ln 5$ Match!

$$\boxed{\begin{aligned} x=0 &\Rightarrow u = 1+e^{0} = 2 \\ x=\ln 3 &\Rightarrow u = 1+e^{2\ln 3} \\ &= 1+\cancel{e}^{\cancel{\ln}(3^2)^{9}} = 10 \end{aligned}}$$

7(c) $\displaystyle\int_{e^3}^{e^8} \frac{1}{x\sqrt{1+\ln x}}\,dx = \int_{4}^{9} \frac{1}{\sqrt{u}}\,du = \int_{4}^{9} u^{-\frac{1}{2}}\,du = \frac{u^{\frac{1}{2}}}{\frac{1}{2}}\Bigg|_{4}^{9} = 2\sqrt{u}\,\Big|_{4}^{9}$

$$\boxed{\begin{aligned} u &= 1+\ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}}$$

$= 2\left(\cancelto{3}{\sqrt{9}} - \cancelto{2}{\sqrt{4}}\right) = 2(3-2) = 2$ Match!

$$\boxed{\begin{aligned} x=e^3 &\Rightarrow u = 1+\cancel{\ln e}^{3} = 4 \\ x=e^8 &\Rightarrow u = 1+\ln e^8 = 9 \end{aligned}}$$