Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 106

Midterm Exam #3

April 18, 2018

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

• Simplify numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $\ln(e^3)$, $e^{2\ln 3}$ and $4^{\frac{3}{2}}$.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		28
2		12
3		8
4		40
5		6
6		6
Total		100

1. [28 Points] Compute each of the following derivatives.

(a) y' where $y = \ln\left(\dfrac{\ln x\,\sqrt{1+e^x}}{(4-x^6)^3\,e^{-\cos x}}\right)$ Do not simplify your final answer here.

$$y = \ln\left(\ln x\,\sqrt{1+e^x}\right) - \ln\left((4-x^6)^3 e^{-\cos x}\right)$$

$-\cos x$

$$= \ln(\ln x) + \ln\sqrt{1+e^x} - \left[\ln\left((4-x^6)^3\right) + \cancel{\ln e}^{-\cos x}\right]$$

$$= \ln(\ln x) + \frac{1}{2}\ln(1+e^x) - 3\ln(4-x^6) + \cos x$$

$$\boxed{y' = \frac{1}{\ln x}\cdot\frac{1}{x} + \frac{1}{2}\cdot\left(\frac{1}{1+e^x}\right)\cdot e^x - 3\left(\frac{1}{4-x^6}\right)(-6x^5) - \sin x}$$

(b) $\dfrac{d}{dx}(\cos x)^{\sin x}$

Let $y = (\cos x)^{\sin x}$

$$\ln y = \ln\left[(\cos x)^{\sin x}\right]$$

$$\ln y = \sin x \ln(\cos x)$$

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\sin x \cdot \ln(\cos x)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \sin x\cdot\frac{1}{\cos x}(-\sin x) + \ln(\cos x)\cos x$$

$$\frac{dy}{dx} = y\left[\frac{-\sin^2 x}{\cos x} + \cos x \ln(\cos x)\right]$$

$$\boxed{= (\cos x)^{\sin x}\left[\frac{-\sin^2 x}{\cos x} + \cos x \ln(\cos x)\right]}$$

Resubstitute

$\ln x + \ln(e^x)$

1. (Continued) Compute each of the following derivatives.

(c) $\dfrac{dy}{dx}$ where $y = \ln(\underbrace{\ln(\cancel{\ln(e^x)})}) + \dfrac{e}{\ln x} + \dfrac{\ln x}{e} + (\ln x \cdot e^x) + \ln(xe^x)$

$\ln(\ln x) + e(\ln x)^{-1} + \frac{1}{e} \cdot \ln x + (\ln x \cdot e^x) + \ln x + x$

$$\frac{dy}{dx} = \boxed{\frac{1}{\ln x} \cdot \frac{1}{x} - e(\ln x)^{-2}\left(\frac{1}{x}\right) + \left(\ln x \cdot e^x + e^x \cdot \left(\frac{1}{x}\right)\right) + \frac{1}{x} + 1}$$

(d) $\dfrac{dy}{dx}$ where $y = \underbrace{\cancel{e}^{\ln(\ln x)}}_{\ln x} + \underbrace{\overbrace{\ln(e^5)}^{\text{constant}}}_{5} + \underbrace{e^{\ln x}}_{x} + \underbrace{\overbrace{5^{\ln e}}^{\text{constant}}}_{5}$

$$\frac{dy}{dx} = \frac{1}{x} + 0 + 1 + 0$$

$$= \boxed{\frac{1}{x} + 1}$$

Note: if don't simplify

$$\frac{dy}{dx} = \underbrace{e^{\ln(\ln x)}}_{\ln x} \cdot \frac{1}{\cancel{\ln x}} \cdot \frac{1}{x} + \cancel{\frac{1}{e^5} \cdot e^5 \cdot 0} + \underbrace{e^{\ln x} \cdot \frac{1}{x}}_{1} + 0$$

cancel Anyhow

2. [12 Points]

(a) Find the Absolute Maximum and/or Minimum Values for the function $f(x) = \frac{x+2}{e^x}$.

$$f'(x) = \frac{e^x(1) - (x+2)e^x}{(e^x)^2} = \frac{\cancel{e^x}(1-x-2)}{e^{2\cancel{x}}} = \frac{-x-1}{e^x} \overset{\text{set}}{=} 0$$

$+$ denominator

$\Rightarrow -x-1=0$

$\Rightarrow x=-1$ Critical #.

Sign Testing into $f'(x)$

	-10	-1	0
f'	$+$		$-$
f	$\nearrow$	Absolute Max	$\searrow$

Max Value $f(-1) = \frac{-1+2}{e^{-1}} = e$

Absolute Maximum Value of $\boxed{e}$ at $x=-1$.

Absolute Minimum Value: NONE

2. (Continued)

(b) At what point on the curve $y = [\ln(x+4)]^2$ is the tangent line horizontal?

$$\frac{dy}{dx} = \frac{2\ln(x+4)}{x+4} \overset{\text{set}}{=} 0$$

$$\Rightarrow 2\ln(x+4) = 0$$

$$\Rightarrow \ln(x+4) = 0.$$

$$\Rightarrow \cancel{e}^{\cancel{\ln}(x+4)} = \cancel{e^0}\ 1$$

$$\Rightarrow x+4 = 1$$

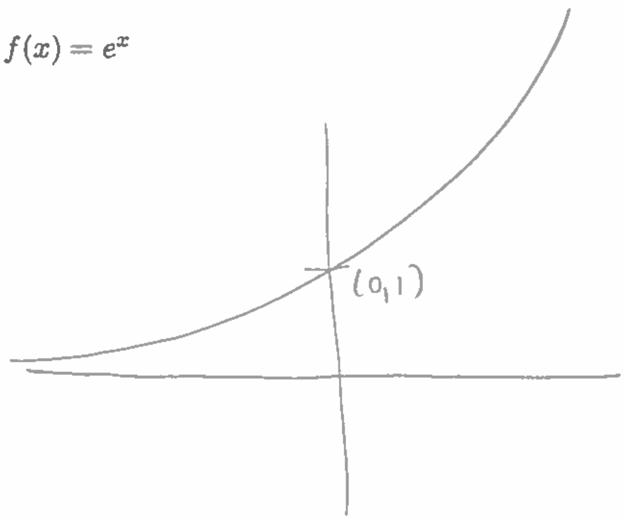
$$\boxed{x = -3}$$

y-value @ $x = -3$ $\Rightarrow$ $y = [\underbrace{\cancel{\ln(-3+4)}}_{\cancel{\ln 1}\ 0}]^2 = 0$

Point. $(-3, f(-3)) = \boxed{(-3, 0)}$

3. [8 Points] Sketch the graphs for each of the following functions. For each, state both the Domain and the Range.

(a) $f(x) = e^x$

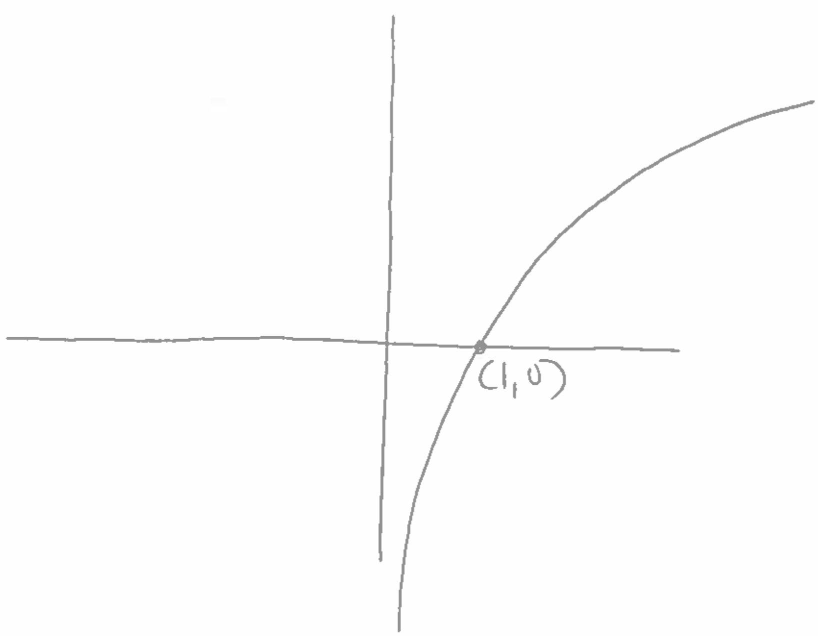


Domain $= \mathbb{R} = (-\infty, \infty)$

Range $= (0, \infty)$

$= \{y \mid y > 0\}$

(b) $f(x) = \ln x$



Domain: $(0, \infty)$

$= \{x \mid x > 0\}$

Range: $(-\infty, \infty)$

$= \mathbb{R}$

4. [40 Points] Evaluate each of the following integrals. Simplify.

(a) $\int \frac{1}{x^3\, e^{\frac{1}{x^2}}}\, dx$ $= -\frac{1}{2}\int \frac{1}{e^u}\, du = -\frac{1}{2}\int e^{-u}\, du$

$u = \frac{1}{x^2} = x^{-2}$

$du = -2x^{-3}\, dx$

$-\frac{1}{2}\, du = \frac{1}{x^3}\, dx$

$= \cancel{-}\frac{1}{2}\frac{e^{-u}}{(\cancel{-}1)} + C$

$= \frac{1}{2e^u} + C$

$= \boxed{\frac{1}{2e^{1/x^2}} + C}$

(b) $\int_1^{\sqrt{6}} \frac{x}{7 - x^2}\, dx$ $= -\frac{1}{2}\int_6^1 \frac{1}{u}\, du = -\frac{1}{2} \ln|u| \Big|_6^1$

$u = 7 - x^2$

$du = -2x\, dx$

$-\frac{1}{2}\, du = x\, dx$

$x = 1 \Rightarrow u = 7 - 1 = 6$

$x = \sqrt{6} \Rightarrow u = 7 - (\sqrt{6})^2 = 7 - 6 = 1$

$= -\frac{1}{2}\left[\cancel{\ln 1}^{\,0} - \ln 6\right]$

$= \boxed{\frac{\ln 6}{2}}$ or $\boxed{\ln\sqrt{6}}$

4. (Continued) Evaluate each of the following integrals. Simplify.

(c) $\int_{e^3}^{e^8} \frac{4}{x\sqrt{1+\ln x}}\,dx = 4\int_4^9 \frac{1}{\sqrt{u}}\,du = \frac{4u^{1/2}}{1/2}\Big|_4^9 = 8\sqrt{u}\Big|_4^9$

$u^{-1/2}$

$\boxed{\begin{array}{l} u = 1+\ln x \\ du = \frac{1}{x}\,dx \end{array}}$ $\boxed{\begin{array}{l} x=e^3 \Rightarrow u = 1+\ln(e^3) = 1+3 = 4 \\ x=e^8 \Rightarrow u = 1+\ln(e^8) = 1+8 = 9 \end{array}}$

$= 8\left[\sqrt{9}^{\,3} - \sqrt{4}^{\,2}\right]$

$= 8(1)$

$= \boxed{8}$

(d) $\int_{-3}^{-1} \frac{1-x}{x^2}\,dx = \int_{-3}^{-1} \frac{1}{x^2} - \frac{1}{x}\,dx = \frac{x^{-1}}{-1} - \ln|x|\Big|_{-3}^{-1}$

x^{-2}

Need $|\cdot|$

$= \frac{-1}{x} - \ln|x|\Big|_{-3}^{-1} = \frac{-1}{(-1)} - \ln 1^{\,0} - \left[\frac{-1}{-3} - \ln 3\right]$

$= 1 - \frac{1}{3} + \ln 3 = \boxed{\frac{2}{3} + \ln 3}$

4. (Continued) Evaluate the following integral. Simplify.

(e) Show that $\int_{\frac{\pi}{18}}^{\frac{\pi}{9}} \tan(3x)\, dx = \boxed{\frac{\ln 3}{6}}$

$$= \int_{\pi/18}^{\pi/9} \frac{\sin(3x)}{\cos(3x)}\, dx = -\frac{1}{3}\int_{\frac{\sqrt{3}}{2}}^{1/2} \frac{1}{u}\, du = -\frac{1}{3} \ln|u| \Big|_{\frac{\sqrt{3}}{2}}^{\frac{1}{2}}$$

$$\boxed{\begin{aligned} u &= \cos(3x) \\ du &= -3\sin(3x)dx \\ -\tfrac{1}{3}du &= \sin(3x)dx \end{aligned}}$$

$$\boxed{\begin{aligned} x = \pi/18 &\Rightarrow u = \cos(\pi/6) = \tfrac{\sqrt{3}}{2} \\ x = \pi/9 &\Rightarrow u = \cos(\pi/3) = 1/2 \end{aligned}}$$

$$= -\frac{1}{3}\left[\ln\left(\frac{1}{2}\right) - \ln\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= -\frac{1}{3}\left[\cancel{\ln 1}^{\,0} - \cancel{\ln 2} - \ln\sqrt{3} + \cancel{\ln 2}\right]$$

cancel

$$= +\frac{\ln\sqrt{3}}{3}$$

$$= \frac{\ln(3^{1/2})}{3} = \frac{1}{2}\left(\frac{1}{3}\right)\ln 3$$

$$= \boxed{\frac{\ln 3}{6}}$$

Match.

Algebra check.

$$(e^x + e^{-2x})(e^x + e^{-2x})$$
$$= e^{2x} + 2e^{-x} + e^{-4x}$$

4. (Continued) Evaluate the following integral. Simplify.

(f) Show that $\displaystyle\int_0^{\ln 2}\left(e^x + \frac{1}{e^{2x}}\right)^2 dx = \boxed{\frac{175}{64}}$

Algebra
$$= \int_0^{\ln 2} e^{2x} + 2e^{-x} + e^{-4x}\,dx$$

$$= \frac{e^{2x}}{2} - 2e^{-x} + \frac{e^{-4x}}{-4}\Big|_0^{\ln 2}$$

(k-rule)
$$\int e^{kx}dx = \frac{e^{kx}}{k} + C$$
k = constant $\neq 0$

$$= \frac{e^{2\ln 2}}{2} - 2e^{-\ln 2} - \frac{e^{-4\ln 2}}{4} - \left(\frac{e^0}{2} - 2e^0 - \frac{e^0}{4}\right)$$

$$= \frac{\cancel{e}^{\cancel{\ln}(2^2)}}{2}\ \frac{-2}{\cancel{e}^{\cancel{\ln} 2}} - \frac{1}{4\cancel{e}^{\cancel{\ln}(2^4)}} - \frac{1}{2} + 2 + \frac{1}{4}$$

16

$$= \frac{4}{2} - \frac{2}{2} - \frac{1}{64} - \frac{1}{2} + 2 + \frac{1}{4}$$

$$= 2 - 1 - \frac{1}{64} - \frac{1}{2} + 2 + \frac{1}{4}$$

$$= 3 - \frac{1}{64} - \frac{1}{4} = \frac{192}{64} - \frac{1}{64} - \frac{16}{64} = \boxed{\frac{175}{64}}$$ Match

$$\begin{array}{r} 64 \\ \times 3 \\ \hline 192 \end{array} \qquad \begin{array}{r} 192 \\ -17 \\ \hline 175 \end{array}$$

$f(x) = \int f'(x)\,dx$

5. [6 Points] Find the function $f(x)$ that satsifies $f'(x) = \dfrac{1}{e^{2x}\left(1-2e^{-2x}\right)^2}$ and $f(0) = -1$.

$$f(x) = \int \frac{1}{e^{2x}(1-2e^{-2x})^2}\,dx = \frac{1}{4}\int \frac{1}{u^2}\,du \qquad u^{-2}$$

$$u = 1-2e^{-2x}$$
$$du = 4e^{-2x}\,dx$$
$$\frac{1}{4}\,du = \frac{1}{e^{2x}}\,dx$$

$$= \frac{1}{4}\left(\frac{u^{-1}}{-1}\right) + C$$

$$= \frac{-1}{4u} + C$$

$$= \frac{-1}{4(1-2e^{-2x})} + C$$

$$f(0) = \frac{-1}{4(1-2\cancel{e^{0}})} + C \overset{set}{=} -1$$

$$1$$

$$-\frac{1}{4(-1)} + C = -1$$

$$\tfrac{1}{4} + C = -1 \Rightarrow C = -\tfrac{5}{4}$$

Finally, $f(x) = \boxed{\dfrac{-1}{4(1-2e^{-2x})} - \dfrac{5}{4}}$

6. [6 Points] Consider $y = \ln x$. **Prove that** $\frac{dy}{dx} = \frac{1}{x}$.

Let $y = \ln x$

Invert $e^y = \cancel{e^{\ln x}} = x$

Differentiate

$$\frac{d}{dx}\left[e^y\right] = \frac{d}{dx}\left[x\right]$$

$$e^y \cdot \frac{dy}{dx} = 1$$

Solve $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ ✓

OR $= \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}$. ✓