Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 106

Midterm Exam #2

March 26, 2014

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

• Simplify numerical answers such as $\sin\left(\frac{\pi}{6}\right)$ and $4^{\frac{3}{2}}$.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		20
2		10
3		25
4		25
5		20
Total		100

1. [20 Points] Compute $\int_1^5 5 - 2x - x^2 \, dx$ using two different methods:

(a) Fundamental Theorem of Calculus

$$\int_1^5 5-2x-x^2 dx = 5x - x^2 - \frac{x^3}{3}\Big|_1^5 = \cancel{25} - \cancel{25} - \frac{125}{3} - (5-1-1/3)$$

$$= \frac{-125}{3} - 4 + \frac{1}{3} = -\frac{124}{3} - 4 = \frac{-124}{3} - \frac{12}{3} = \boxed{\frac{-136}{3}}$$

(b) Limit Definition of the Definite Integral.

$a=1 \quad b=5 \quad f(x) = 5-2x-x^2$

$\Delta x = \frac{b-a}{n} = \frac{5-1}{n} = \frac{4}{n} \quad x_i = a + i\Delta x = 1 + i\left(\frac{4}{n}\right) = 1 + \frac{4i}{n}$

$$= \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n f(1+4i/n)\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^n 5 - 2\left[1+\frac{4i}{n}\right] - \left[1+\frac{4i}{n}\right]^2$$

$$= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^n 5 - 2 - \frac{8i}{n} - 1 - \frac{8i}{n} - \frac{16i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^n 2 - \frac{16i}{n} - \frac{16i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^n 2 - \frac{4}{n}\sum_{i=1}^n \frac{16i}{n} - \frac{4}{n}\sum_{i=1}^n \frac{16i^2}{n^2}$$

$$= \lim_{n\to\infty} \frac{8}{n}\sum_{i=1}^n 1 - \frac{64}{n^2}\sum_{i=1}^n i - \frac{64}{n^3}\sum_{i=1}^n i^2$$

$$= \lim_{n\to\infty} \frac{8}{n}(n) - \frac{64}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{64}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} 8 - \frac{64}{2}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) - \frac{64}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$$

$$= \lim_{n\to\infty} 8 - 32(1)(1+\cancel{1/n}^{0}) - \frac{32}{3}(1)(1+\cancel{1/n}^{0})(2+\cancel{1/n}^{0})$$

$$= 8 - 32 - \frac{64}{3} = -24 - \frac{64}{3} = -\frac{72}{3} - \frac{64}{3} = \boxed{\frac{-136}{3}}$$

Match!

2. [10 Points] Compute $g'(x)$ where $g(x) = \int_x^4 \frac{1-\sin t}{t^2+\tan t+\frac{9}{t}}\,dt.$ $= -\int_4^x \frac{1-\sin t}{t^2+\tan t+9/t}\,dt$

$$g'(x) = -\frac{d}{dx}\int_4^x \frac{1-\sin t}{t^2+\tan t+9/t}\,dt = \boxed{-\left[\frac{1-\sin x}{x^2+\tan x+9/x}\right]}$$

FTC Part I

Note: cosine even function

$\Rightarrow \cos(-\pi/6) = \cos(\pi/6)$

3. [25 Points] Evaluate each of the following integrals. Simplify.

(a) $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\left(\frac{x}{2}\right)\, dx = 2\int_{-\pi/6}^{\pi/4} \sin u \, du = -2\cos u \Big|_{-\pi/6}^{\pi/4}$

$u = \frac{x}{2}$

$du = \frac{1}{2} dx$

$2du = dx$

$x = -\pi/3 \Rightarrow u = -\pi/6$

$x = \pi/2 \Rightarrow u = \pi/4$

$= -2\left[\cos(\pi/4) - \cos(-\pi/6)\right]$

$\sqrt{2}/2 \qquad \sqrt{3}/2$

$= -2\left[\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\right]$

$= \boxed{\sqrt{3} - \sqrt{2}}$

(b) $\displaystyle\int \frac{\sqrt{2}\,\sec^2(3x+4)}{\tan^2(3x+4)}\, dx = \frac{\sqrt{2}}{3}\int \frac{1}{u^2}\, du = \frac{\sqrt{2}}{3}\int u^{-2} du = \frac{\sqrt{2}}{3}\left(\frac{u^{-1}}{(-1)}\right) + C$

$u = \tan(3x+4)$

$du = \sec^2(3x+4)(3)\,dx$

$\frac{1}{3} du = \sec^2(3x+4)\,dx$

$= \frac{-\sqrt{2}}{3u} + C$

$= \boxed{\frac{-\sqrt{2}}{3\tan(3x+4)} + C}$

3. (Continued) Evaluate each of the following integrals. Simplify.

(c) $\displaystyle\int_{\frac{\pi^2}{4}}^{\pi^2} \frac{\cos\sqrt{x}}{\sqrt{x}\,(1+\sin\sqrt{x}\,)^3}\,dx = 2\int_2^1 \frac{1}{u^3}\,du = 2\int_2^1 u^{-3}\,du = \frac{\cancel{2}u^{-2}}{-2}\Bigg|_2^1$

$$\boxed{\begin{aligned} u &= 1+\sin\sqrt{x} \\ du &= \cos\sqrt{x}\left(\frac{1}{2\sqrt{x}}\right)dx \\ 2\,du &= \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx \end{aligned}}$$

⟨watch order on limits⟩

$$= \frac{-1}{u^2}\Bigg|_2^1 = -1-\left(-\tfrac{1}{4}\right) = \boxed{\frac{-3}{4}}$$

$$\boxed{\begin{aligned} x &= \pi^2/4 \Rightarrow u = 1+\overset{1}{\sin \pi/2} = 2 \\ x &= \pi^2 \Rightarrow u = 1+\overset{0}{\sin\pi} = 1 \end{aligned}}$$

(d) $\displaystyle\int \frac{\cos x+\sin x}{\sqrt{\cos x-\sin x}}\,dx = -\int \frac{1}{\sqrt{u}}\,du = -\int u^{-1/2}\,du = -2\sqrt{u}+C$

$$= \boxed{-2\sqrt{\cos x-\sin x}+C}$$

$$\boxed{\begin{aligned} u &= \cos x-\sin x \\ du &= -\sin x-\cos x\,dx \\ -du &= \sin x+\cos x\,dx \end{aligned}}$$

$\frac{7}{4} - \frac{1}{2} = \frac{7}{4} - \frac{2}{4} = \frac{5}{4}$ $\qquad$ $-\frac{1}{3} - \frac{1}{2} = -\frac{2}{6} - \frac{3}{6} = -\frac{5}{6}$

4. [25 Points] Evaluate each of the following integrals. Simplify.

(a) $\displaystyle\int \frac{x^{\frac{7}{4}} + x^{-\frac{1}{3}}}{\sqrt{x}}\, dx = \int \frac{x^{7/4}}{\sqrt{x}} + \frac{x^{-1/3}}{\sqrt{x}}\, dx = \int x^{5/4} + x^{-5/6}\, dx$

$$= \boxed{\frac{4}{9} x^{9/4} + 6x^{1/6} + C}$$

(b) $\displaystyle\int \frac{5}{x^2\left(5 + \frac{3}{x}\right)^{\frac{5}{3}}}\, dx = -\frac{5}{3}\int \frac{1}{u^{5/3}}\, du = -\frac{5}{3}\int u^{-5/3}\, du$

$$= -\frac{5}{3}\,\frac{u^{-2/3}}{\left(-\frac{2}{3}\right)} + C$$

$$= -\frac{5}{\cancel{3}}\left(-\frac{\cancel{3}}{2}\right) u^{-2/3} + C$$

$$= \boxed{\frac{5}{2\left(5+\frac{3}{x}\right)^{2/3}} + C}$$

$$\boxed{\begin{aligned} u &= 5 + \tfrac{3}{x} \\ du &= -\tfrac{3}{x^2}\, dx \\ -\tfrac{1}{3}\, du &= \tfrac{1}{x^2}\, dx \end{aligned}}$$

4. (Continued) Evaluate each of the following integrals. Simplify.

(c) $\int_{-2}^{-1} \left(x - \frac{5}{x^3}\right)^2 dx = \int_{-2}^{-1} x^2 - \frac{10}{x^2} + \frac{25}{x^6}\, dx$

Algebra

$$= \frac{x^3}{3} + \frac{10}{x} - \frac{5}{x^5} \Big|_{-2}^{-1}$$

$$= \frac{(-1)^3}{3} + \frac{10}{(-1)} - \frac{5}{(-1)^5} - \left(\frac{(-2)^3}{3} + \frac{10}{(-2)} - \frac{5}{(-2)^5}\right)$$

$$= -\tfrac{1}{3} \cancel{-10} \cancel{+5} + \tfrac{8}{3} \cancel{+5} - \frac{5}{32}$$

$$= \tfrac{7}{3} - \frac{5}{32} = \frac{224}{96} - \frac{15}{96} = \boxed{\frac{209}{96}}$$

(d) $\int_{-3}^{-2} x\,(x+2)^7\, dx = \int_{-1}^{0} (u-2)u^7 du = \int_{-1}^{0} u^8 - 2u^7\, du$

invert

$\boxed{u = x+2 \Rightarrow x = u-2}$

$\boxed{du = dx}$

$\boxed{x=-3 \Rightarrow u=-1}$

$\boxed{x=-2 \Rightarrow u=0}$

$$= \frac{u^9}{9} - \frac{\cancel{2}u^8}{\cancel{8}_{4}} \Big|_{-1}^{0}$$

$$= (0 - 0) - \left(\frac{(-1)^9}{9} - \frac{(-1)^8}{4}\right)$$

$$= -\left(-\tfrac{1}{9} - \tfrac{1}{4}\right)$$

$$= \tfrac{1}{9} + \tfrac{1}{4}$$

$$= \frac{4}{36} + \frac{9}{36} = \boxed{\frac{13}{36}}$$

5. [20 Points] Consider an object travelling with velocity $v(t) = 3t - 9$ meters per second.

(a) Compute the **displacement** for the object from time $t = 1$ to $t = 4$.

$$\text{Displacement} = \int_1^4 v(t)\,dt = \int_1^4 3t-9\,dt = \frac{3t^2}{2} - 9t \Big|_1^4$$

$$= (24-36) - (3/2 - 9)$$

$$= -12 - 3/2 + 9 = -3 - 3/2 = \frac{-6}{2} - 3/2 = \boxed{\frac{-9}{2}}$$

(b) Compute the **total distance** travelled by the object from time $t = 1$ to $t = 4$.

$$\text{Total Distance} = \int_1^4 |v(t)|\,dt = \int_1^4 |3t-9|\,dt$$

$$= \int_1^3 |3t-9|\,dt + \int_3^4 |3t-9|\,dt$$

$$|3t-9| = \begin{cases} 3t-9 & \text{if } t \geq 3 \\ -(3t-9) & \text{if } t < 3 \end{cases}$$

$$= \int_1^3 -3t+9\,dt + \int_3^4 3t-9\,dt$$

$$= \frac{-3t^2}{2} + 9t \Big|_1^3 + \frac{3t^2}{2} - 9t \Big|_3^4$$

$$= \left(\frac{-27}{2} + 27\right) - \left(-3/2 + 9\right) + (24-36) - \left(\frac{27}{2} - 27\right)$$

$$= \frac{-27}{2} + 27 + 3/2 - 9 - 12 - \frac{27}{2} + 27$$

$$54 \quad -21 \quad 33$$

$$= \frac{-51}{2} + 33$$

$$= \frac{-51}{2} + \frac{66}{2} = \boxed{\frac{15}{2}}$$

Should be +.