Name: Me

Amherst College

DEPARTMENT OF MATHEMATICS

Math 106

Midterm Exam #1

February 15, 2017

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

• Simplify numerical answers such as $\sin\left(\frac{\pi}{6}\right)$ and $4^{\frac{3}{2}}$.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		22
2		20
3		12
4		16
5		20
6		10
Total		100

1. [22 Points] Differentiate each of the following functions. **Do not** simplify your answers.

(a) $f(x) = \sqrt{\sin x}$

$f'(x) = \boxed{\frac{1}{2\sqrt{\sin x}} \cdot \cos x}$

(b) $f(x) = \sin\sqrt{x}$

$f'(x) = \boxed{\cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}}}$

(c) $f(x) = \sin^2(\tan x)$ $= [\sin(\tan x)]^2$

$f'(x) = \boxed{2\sin(\tan x) \cdot \cos(\tan x) \cdot \sec^2 x}$

(d) $f(x) = \sin x \cdot \tan\left(\frac{7}{x^6}\right)$ $7x^{-6}$

$f'(x) = \boxed{\sin x \cdot \sec^2\left(\frac{7}{x^6}\right)(-42x^{-7}) + \tan\left(\frac{7}{x^6}\right) \cdot \cos x}$

(e) $f(x) = \left(\frac{\cos(7x)}{x^2 + \sec x}\right)^{\frac{7}{8}}$

$f'(x) = \boxed{\frac{7}{8}\left(\frac{\cos(7x)}{x^2+\sec x}\right)^{-1/8}\left[\frac{(x^2+\sec x)(-\sin(7x))7 - \cos(7x)(2x+\sec x\tan x)}{(x^2+\sec x)^2}\right]}$

2. [20 Points]

(a) Let $f(x) = \dfrac{1}{\tan^2 x} + \cos^2 x + \sec(2x)$. Compute $f'\left(\frac{\pi}{6}\right)$. Simplify.

$= \tan^{-2} x + \cdots$

$$f'(x) = -2\tan^{-3}x \cdot \sec^2 x + 2\cos x(-\sin x) + \sec(2x)\tan(2x)(2)$$

$\cos \pi/6 = \sqrt{3}/2$

$\cos \pi/3 = 1/2$

$\sec \pi/3 = 2.$

$$f'(\pi/6) = \frac{-2\sec^2(\pi/6)}{\tan^3(\pi/6)} - 2\cos\left(\tfrac{\pi}{6}\right)\sin\left(\tfrac{\pi}{6}\right) + 2\sec\left(\tfrac{\pi}{3}\right)\tan\left(\tfrac{\pi}{3}\right)$$

$\frac{\sqrt{3}}{2}$ $\quad \frac{1}{2}$ $\quad 2 \cdot 2\sqrt{3}$

$$= -2\frac{1}{\left(\frac{1}{\sqrt{3}}\right)^3}\cdot\left(\frac{2}{\sqrt{3}}\right)^2 - \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)\left(\tfrac{1}{2}\right) + 4\sqrt{3}$$

$$= -2(\cancel{3}\sqrt{3})\cdot\frac{4}{\cancel{3}} - \frac{\sqrt{3}}{2} + 4\sqrt{3}$$

$$= -8\sqrt{3} - \frac{\sqrt{3}}{2} + 4\sqrt{3} = -4\sqrt{3} - \frac{\sqrt{3}}{2} = -\frac{8\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = \boxed{\frac{-9\sqrt{3}}{2}}$$

(b) Let $f(x) = 4\sin\left(x - \frac{\pi}{4}\right) - \cos x - \tan^2 x$. Show that $f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$.

$$f'(x) = 4\cos(x - \pi/4) + \sin x - 2\tan x\sec^2 x$$

$\cos \pi/4 = 1/\sqrt{2}$

$\sec \pi/4 = \sqrt{2}$

$$f'(\pi/4) = 4\cos 0 + \sin(\pi/4) - 2\tan(\pi/4)\cdot\sec^2(\pi/4)$$

1

$$= \cancel{4} + \frac{\sqrt{2}}{2} - \underbrace{2(1)(\sqrt{2})^2}_{\cancel{-4}} = \boxed{\frac{\sqrt{2}}{2}} \checkmark$$

3. [12 Points] Let $f(x) = \frac{5}{6}x + x^{\frac{5}{6}} + \frac{1}{x^{\frac{5}{6}}} + \frac{6}{5} + \frac{5}{6x^6} - \frac{6}{x^5}$.

Compute the most general antiderivative

$\int f(x)\,dx = \int \frac{5}{6}x + x^{\frac{5}{6}} + \frac{1}{x^{\frac{5}{6}}} + \frac{6}{5} + \frac{5}{6x^6} - \frac{6}{x^5}\,dx$ Do not simplify.

$$= \int \frac{5}{6}x + x^{5/6} + x^{-5/6} + \frac{6}{5} + \frac{5}{6}x^{-6} - 6x^{-5}dx$$

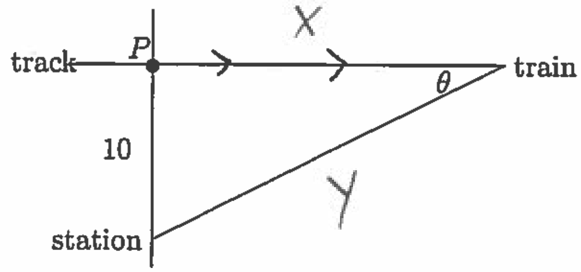
$$= \frac{5}{6}\cdot\frac{x^2}{2} + \frac{x^{11/6}}{11/6} + \frac{x^{1/6}}{1/6} + \frac{6}{5}x + \cancel{\frac{5}{6}}\cdot\frac{x^{-5}}{\cancel{-5}} - 6\frac{x^{-4}}{-4} + C$$

$$= \boxed{\frac{5x^2}{12} + \frac{6}{11}x^{11/6} + 6x^{1/6} + \frac{6}{5}x - \frac{1}{6x^5} + \frac{3}{2x^4} + C}$$

4. [16 Points] Consider a point P on a train track. Suppose a train depot station is 10 feet directly south from this point P. The train is travelling east at 6 feet per second. Consider the angle as shown in the diagram. How fast is this angle changing when 2 seconds has passed since the train passed point P.

- Diagram

The picture at arbitrary time t is:



- Variables

Let x = distance the train has travelled East past point P.

y = distance between train and the station

Given $\frac{dx}{dt} = 6$ ft/sec. Find $\frac{d\theta}{dt} = ?$ when $x = 6\text{ft}/\cancel{\text{sec}}\,(2\cancel{\text{sec}}) = 12$ feet.

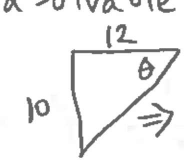
- Equation

$$\tan\theta = \frac{10}{x} = 10x^{-1}$$

- Differentiate

$$\sec^2\theta \cdot \frac{d\theta}{dt} = -\frac{10}{x^2} \cdot \frac{dx}{dt}$$ Related Rates!

- Extra Solvable Information



$\Rightarrow y = \sqrt{(10)^2 + (12)^2} = \sqrt{100+144} = \sqrt{244}$ $\Rightarrow \sec\theta = \frac{H}{A} = \frac{\sqrt{244}}{12}$

- Substitute

$$\left(\frac{\sqrt{244}}{12}\right)^2 \frac{d\theta}{dt} = \frac{-10}{(12)^2} \cdot 6$$

- Solve $\frac{d\theta}{dt} = \frac{-10}{\cancel{144}} \cdot 6 \cdot \frac{\cancel{144}}{244} = \frac{-60}{244} = \frac{-15}{61}$ Rad./sec.

- Answer: The angle is shrinking at $15/61$ Radians per second, at that time

5. [20 Points]

(a) Consider a function f such that $f'(x) = \dfrac{x^{\frac{1}{5}} + x^{-\frac{2}{3}}}{x^{\frac{2}{3}}}$. Compute $f(x)$.

$$f'(x) = x^{-2/3}\left(x^{1/5} + x^{-2/3}\right) = x^{-2/3} \cdot x^{1/5} + x^{-4/3} = x^{-10/15} \cdot x^{3/15} + x^{-4/3}$$

$$= x^{-7/15} + x^{-4/3}.$$

Antidifferentiate

$$f(x) = \frac{x^{8/15}}{8/15} + \frac{x^{-1/3}}{-1/3} + C$$

$$= \boxed{\frac{15}{8} x^{8/15} - 3x^{-1/3} + C}$$

(b) Consider a function f such that $f''(x) = \pi \sin x + 2\cos x$ and $f'\left(\frac{\pi}{2}\right) = 0$ and $f(\pi) = 2$. Compute $f(x)$.

Antidifferentiate:

$$f'(x) = -\pi\cos x + 2\sin x + C_1$$

$$f'(\pi/2) = -\pi \underbrace{\cos \pi/2}_{0} + 2\underbrace{\sin \pi/2}_{1} + C_1 \overset{\text{set}}{=} 0.$$

$$2 + C_1 = 0 \Rightarrow C_1 = -2$$

Clean Up:

$$f'(x) = -\pi\cos x + 2\sin x - 2$$

Antidifferentiate Again:

$$f(x) = -\pi\sin x - 2\cos x - 2x + C_2$$

$$f(\pi) = -\pi \underbrace{\sin \pi}_{0} - 2\underbrace{\cos \pi}_{(-1)} - 2\pi + C_2 \overset{\text{set}}{=} 2$$

$$\cancel{2} - 2\pi + C_2 = \cancel{2} \Rightarrow C_2 = 2\pi$$

Finally, $f(x) = \boxed{-\pi\sin x - 2\cos x - 2x + 2\pi}$

6. [10 Points] A ball is thrown upwards from the top of a building with an initial *speed* of 32 feet per second. The ball hits the ground below with a *speed* of 64 feet per second. How tall is the building?

Hint: Use a(t) = -32 feet per second squared as acceleration due to gravity on the falling body.

$a(t) = -32$

$v(t) = -32t + v_0 = -32t + 32$

$s(t) = -16t^2 + 32t + s_0$

$\underset{\text{impact}}{v(t)} = -32t + 32 = -64$ Down

$-32t = -96$

$t = 3$ seconds Impact

$s(t_{impact}) = s(3) = -16(3)^2 + 32(3) + s_0 \overset{\text{set}}{=} 0$ Ground.

$= -144 + 96 + s_0 = 0.$

$\Rightarrow s_0 = 144 - 96.$

$= 48$ feet.

$\begin{array}{r} 16 \\ 9 \\ \hline 144 \end{array}$ $\begin{array}{r} 32 \\ 3 \\ \hline 96 \end{array}$

$\boxed{\text{The building is 48 feet Tall.}}$