Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 106 Final Examination

May 7, 2018

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

• You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, or $e^{3\ln 3}$ should be simplified.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		25
2		8
3		56
4		20
5		15
6		12
7		14
8		15
9		15
10		10
11		10
Total		200

1. [25 Points] Compute each of the following derivatives.

(a) $g'(x)$, where $g(x) = (\cos x)^{3x}$. Simplify.

Let $y = (\cos x)^{3x}$

$\ln y = \ln\left[(\cos x)^{3x}\right] = 3x \ln(\cos x)$

$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[3x \ln(\cos x)\right]$

$\frac{1}{y} \cdot \frac{dy}{dx} = 3x \cdot \frac{1}{\cos x}(-\sin x) + \ln(\cos x) \cdot 3$

y

$\frac{dy}{dx} = \boxed{(\cos x)^{3x}\left[-3x\tan x + 3\ln(\cos x)\right]}$

(b) $\frac{d}{dx} \ln\left(\frac{(5-x^2)^9 \sqrt{1+\tan x}}{e^{-\cos x} \cdot \ln x}\right)$ Do not simplify the final answer here.

$= \frac{d}{dx}\left[\ln\left((5-x^2)^9 \cdot \sqrt{1+\tan x}\right) - \ln\left(e^{-\cos x} \cdot \ln x\right)\right]$

$-\cos x$

$= \frac{d}{dx}\left[\ln\left[(5-x^2)^9\right] + \ln\sqrt{1+\tan x}\right] - \left[\cancel{\ln e}^{-\cos x} + \ln(\ln x)\right]$

$= \frac{d}{dx}\left[9\ln(5-x^2) + \frac{1}{2}\ln(1+\tan x) + \cos x - \ln(\ln x)\right]$

$= \boxed{\frac{9}{5-x^2} \cdot (-2x) + \frac{1}{2}\left(\frac{1}{1+\tan x}\right) \cdot \sec^2 x - \sin x - \frac{1}{\ln x} \cdot \left(\frac{1}{x}\right)}$

1. (Continued) Compute each of the following derivatives.

(c) $f'\left(\frac{\pi}{6}\right)$ where $f(x) = \dfrac{1}{2\tan^2 x} + \cos^2 x + \sec(2x)$. Simplify.

$$= \frac{1}{2}(\tan x)^{-2} + (\cos x)^2 + \sec(2x)$$

$$f'(x) = \frac{1}{\cancel{2}}(-\cancel{2})(\tan x)^{-3} \cdot \sec^2 x + 2\cos x(-\sin x) + \sec(2x)\tan(2x) \cdot 2$$

$$f'(\pi/6) = \frac{-1}{[\underbrace{\tan(\pi/6)}_{\frac{1}{\sqrt{3}}}]^3} \cdot \underbrace{\sec^2\left(\frac{\pi}{6}\right)}_{\left(\frac{2}{\sqrt{3}}\right)^2} - 2\overbrace{\cos(\pi/6)}^{\sqrt{3}/2}\overbrace{\sin(\pi/6)}^{1/2} + 2\overbrace{\sec(\pi/3)}^{2}\overbrace{\tan(\pi/3)}^{\sqrt{3}}$$

$$= -(\sqrt{3})^3 \cdot \frac{4}{3} - \cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right)\left(\frac{1}{2}\right) + 2(2)\sqrt{3}$$

$$= -\cancel{3}\sqrt{3} \cdot \frac{4}{\cancel{3}} - \frac{\sqrt{3}}{2} + 4\sqrt{3}$$

$$= \cancel{-4\sqrt{3}} - \frac{\sqrt{3}}{2} + \cancel{4\sqrt{3}} = \boxed{\frac{-\sqrt{3}}{2}}$$

(d) $f'(x)$, where $f(x) = \dfrac{1}{\sqrt{\ln x}} + \dfrac{1}{\ln\sqrt{x}}$. Do not simplify.

$$= (\ln x)^{-1/2} + (\ln\sqrt{x})^{-1}$$

OR $\dfrac{1}{\ln\sqrt{x}} = [\ln\sqrt{x}]^{-1} = \left(\frac{1}{2}\ln x\right)^{-1}$

$= 2(\ln x)^{-1}$

⋮

$$f'(x) = \boxed{-\frac{1}{2}(\ln x)^{-3/2} \cdot \frac{1}{x} - (\ln\sqrt{x})^{-2} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}}$$

2. [8 Points]

(a) Prove that $\frac{d}{dx} \ln x = \frac{1}{x}$.

Let $y = \ln x$

Invert $e^y = \cancel{e}^{\cancel{\ln x}}$ x

Differentiate $\frac{d}{dx}[e^y] = \frac{d}{dx}[x]$

$e^y \cdot \frac{dy}{dx} = 1$

Solve $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ ✓

(b) Let $y = 5^x$. Prove that $\frac{dy}{dx} = 5^x(\ln 5)$

$y = 5^x$

$\ln y = \ln(5^x) = x \cdot (\ln 5)$

$\frac{d}{dx}[\ln y] = \frac{d}{dx}[\underbrace{(\ln 5)}_{\text{constant}} x]$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln 5$

$\frac{dy}{dx} = y \cdot (\ln 5)$

$= 5^x(\ln 5)$ ✓

3. [56 Points] Compute each of the following integrals.

(a) $\int \frac{1}{\sqrt{x}\sqrt{2+\sqrt{x}}}\,dx = 2\int \frac{1}{\sqrt{u}}\,du = 2\int u^{-1/2}\,du = 2\,\frac{u^{1/2}}{1/2} + C$

$$\boxed{\begin{aligned} u &= 2+\sqrt{x} \\ du &= \frac{1}{2\sqrt{x}}\,dx \\ 2du &= \frac{1}{\sqrt{x}}\,dx \end{aligned}}$$

$= 4\sqrt{u} + C$

$= \boxed{4\sqrt{2+\sqrt{x}} + C}$

(b) Show that $\int_0^{\frac{\pi}{6}} \tan(2x)\,dx = \ln\sqrt{2}$

$\int_0^{\pi/6} \tan(2x)\,dx = \int_0^{\pi/6} \frac{\sin(2x)}{\cos(2x)}\,dx = -\frac{1}{2}\int_1^{1/2} \frac{1}{u}\,du = -\frac{1}{2}\ln|u| \Big|_1^{1/2}$

$$\boxed{\begin{aligned} u &= \cos(2x) \\ du &= -\sin(2x)\cdot 2dx \\ -\tfrac{1}{2}\,du &= \sin(2x)\,dx \end{aligned}}$$

$$\boxed{\begin{aligned} x = 0 &\Rightarrow u = \cos 0 = 1 \\ x = \pi/6 &\Rightarrow u = \cos(\pi/3) = \tfrac{1}{2} \end{aligned}}$$

$= -\frac{1}{2}\left[\ln\left(\frac{1}{2}\right) - \cancel{\ln 1}^{\,0}\right]$

$= -\frac{1}{2}\left[\cancel{\ln 1}^{\,0} - \ln 2\right]$

$= + \frac{1}{2}\ln 2 = \ln\left[2^{1/2}\right] = \boxed{\ln\sqrt{2}}$

3. (Continued) Compute each of the following integrals.

(c) $\int \frac{1}{e^x(1-e^{-x})^{\frac{4}{3}}}\,dx = \int \frac{1}{u^{4/3}}\,du = \int u^{-4/3}\,du = \frac{u^{-1/3}}{-1/3} + C$

$\boxed{\begin{aligned} u &= 1-e^{-x} \\ du &= e^{-x}dx \\ &= \frac{1}{e^x}dx \end{aligned}}$

$= \boxed{\frac{-3}{(1-e^{-x})^{1/3}} + C}$

(d) $\int_0^{10} 3 - |x-1|\,dx$

$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \to x \geq 1 \\ -(x-1) & \text{if } x-1 < 0 \to x < 1 \end{cases}$

$= \int_0^1 3-|x-1|\,dx + \int_1^{10} 3-|x-1|\,dx$

$= \int_0^1 3 \overset{+}{-} (-(x-1))\,dx + \int_1^{10} 3-(x-1)\,dx$

$= \int_0^1 2+x\,dx + \int_1^{10} 4-x\,dx$

$= 2x + \frac{x^2}{2}\Big|_0^1 + 4x - \frac{x^2}{2}\Big|_1^{10}$

$= 2 + \frac{1}{2} - (0+0) + 40 - \cancel{\frac{100}{2}}^{-50} - (4-\frac{1}{2})$

$= 2 + \frac{1}{2} + \underbrace{\underbrace{40-50}_{-10} - 4}_{-14} + \frac{1}{2} = -12 + 1 = \boxed{-11}$

3. (Continued) Compute the following integral.

(e) Show that $\int_{-1}^{2} \frac{\overset{x^2 \cdot x}{x^3}}{x^2-5}\,dx = \frac{3}{2} - \ln(32)$

$u = x^2 - 5 \Rightarrow x^2 = u+5$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$x = -1 \Rightarrow u = 1-5 = -4$

$x = 2 \Rightarrow u = 4-5 = -1$

Split

$$= \frac{1}{2}\int_{-4}^{-1} \frac{u+5}{u}\,du$$

$$= \frac{1}{2}\int_{-4}^{-1} 1 + \frac{5}{u}\,du$$

$$= \frac{1}{2}\left[u + 5\ln|u|\right]\Big|_{-4}^{-1}$$

$$= \frac{1}{2}\left[-1 + \cancel{5\ln 1}^{\,0} - (-4 + 5\ln 4)\right]$$

$$= \frac{1}{2}\left[-1 + 4 - 5\ln 4\right]$$

$$= \frac{1}{2}\left[3 - 5\ln 4\right]$$

$$= \frac{3}{2} - \frac{5}{2}\ln 4 = \frac{3}{2} - \ln\left[4^{5/2}\right]$$

$$= \frac{3}{2} - \ln\left[\left(\cancel{\sqrt{4}}^{\,2}\right)^5\right]^{32} = \frac{3}{2} - \ln(2^5) = \boxed{\frac{3}{2} - \ln(32)}$$

3. (Continued) Compute each of the following integrals.

(f) $\int_2^6 \frac{1}{x^2}\cos\left(\frac{\pi}{x}\right)\,dx = -\frac{1}{\pi}\int_{\pi/2}^{\pi/6} \cos u\,du = \frac{-1}{\pi}\sin u\Big|_{\pi/2}^{\pi/6}$

$u = \pi/x$

$du = -\pi/x^2\,dx$

$\frac{-1}{\pi}du = \frac{1}{x^2}dx$

$x=2 \Rightarrow u = \pi/2$

$x=6 \Rightarrow u = \pi/6$

$= \frac{-1}{\pi}\left[\cancelto{\frac{1}{2}}{\sin\left(\frac{\pi}{6}\right)} - \cancelto{1}{\sin\left(\pi/2\right)}\right]$

$= \frac{-1}{\pi}\left[-\frac{1}{2}\right] = \boxed{\frac{1}{2\pi}}$

(g) $\int_1^{e^3} \frac{\sqrt{4-\ln x}}{x}\,dx = -\int_4^1 \sqrt{u}^{\,\to u^{1/2}}\,du = -\frac{2}{3}u^{3/2}\Big|_4^1$

$u = 4 - \ln x$

$du = -\frac{1}{x}dx$

$-du = \frac{1}{x}dx$

$x=1 \Rightarrow u = 4 - \cancelto{0}{\ln 1} = 4$

$x = e^3 \Rightarrow u = 4 - \ln e^3 = 4-3 = 1$

$= -\frac{2}{3}\left[\cancelto{1}{1^{3/2}} - 4^{3/2}\right] = -\frac{2}{3}\left[1 - (\cancelto{2}{\sqrt{4}})^3\right]$

$= -\frac{2}{3}\left[1-8\right]$

$= -\frac{2}{3}(-7) = \boxed{\frac{14}{3}}$

4. [20 Points] Consider the function given by

$$f(x) = \frac{e^{9x}}{9} + e^9 + \frac{e^9}{x} - \frac{1}{9e^x} + 9x^e + \frac{9}{e^{9x}} + \frac{e}{x^9} + \frac{e^x}{e^{9x}} + e^x \cdot e^{9x} + e^{9-x}$$

(a) Compute the **derivative**, $f'(x)$.

PREP: $f(x) = \frac{e^{9x}}{9} + \underset{\text{constant}}{e^9} + e^9 \cdot x^{-1} - \frac{1}{9} e^{-x} + 9x^e + 9e^{-9x} + ex^{-9} + e^{-8x} + e^{10x} + e^{9-x}$

$f'(x) = \frac{1}{9} \cdot e^{9x} \cdot 9 + 0 - e^9 x^{-2} + \frac{1}{9} e^{-x} + 9ex^{e-1} - 81e^{-9x} - 9ex^{-10} - 8e^{-8x} + 10e^{10x} + e^{9-x}(-1)$

(b) Compute the **antiderivative**, $\int f(x)\, dx$.

$\int f(x)dx = \frac{e^{9x}}{81} + e^9 \cdot x + e^9 \ln|x| + \frac{1}{9} e^{-x} + \frac{9x^{e+1}}{e+1} + \frac{9e^{-9x}}{-9} + \frac{ex^{-8}}{-8} + \frac{e^{-8x}}{-8} + \frac{e^{10x}}{10} - e^{9-x} + C$

5. [15 Points] Compute $\int_{-1}^{2} 2 - 2x - x^2 \, dx$ using each of the following two different methods:

(a) Fundamental Theorem of Calculus,
(b) The limit definition of the definite integral.

$a = -1 \quad b = 2 \qquad x_i = a + i\Delta x$

$\Delta x = \frac{b-a}{n} = \frac{2-(-1)}{n} = \frac{3}{n} \qquad = -1 + \frac{3i}{n}$

Recall

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} 1 = n$$

(a) $\int_{-1}^{2} 2 - 2x - x^2 dx = 2x - x^2 - \frac{x^3}{3}\Big|_{-1}^{2} = \cancel{4} - \cancel{4} - \frac{8}{3} - \left(-2 - 1 + \frac{1}{3}\right) = \frac{-8}{3} + 2 + 1 - \frac{1}{3} = -\frac{\cancel{9}^{-3}}{\cancel{3}} + 3 = \boxed{0}$

(b) $\int_{-1}^{2} 2 - 2x - x^2 dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 2 - 2\left(-1 + \frac{3i}{n}\right) - \left(-1 + \frac{3i}{n}\right)^2$

$-\left(1 - 6i/n + 9i^2/n^2\right)$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 2 + 2 - \cancel{\frac{6i}{n}} - 1 + \cancel{\frac{6i}{n}} - \frac{9i^2}{n^2} = \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 3 - \frac{9i^2}{n^2}$

$= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} 3 - \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2} = \lim_{n\to\infty} \frac{9}{n} \sum_{i=1}^{n} 1 - \frac{27}{n^3} \sum_{i=1}^{n} i^2$

$n \qquad \frac{n(n+1)(2n+1)}{6}$

$= \lim_{n\to\infty} \frac{9}{\cancel{n}}(\cancel{n}) - \frac{27}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] = \lim_{n\to\infty} 9 - \frac{27}{6}\left(\frac{\cancel{n}}{\cancel{n}}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$

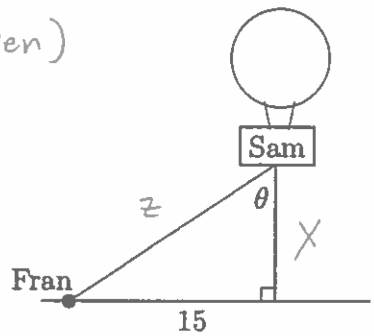
$= \lim_{n\to\infty} 9 - \frac{27}{6}(1)\left(1 + \cancel{\frac{1}{n}}^{0}\right)\left(2 + \cancel{\frac{1}{n}}^{0}\right) = 9 - \cancel{\frac{27}{3}}^{9} = 9 - 9 = \boxed{0}$

$1 \qquad 2$

MATCH!

6. [12 Points] Fran is standing on the ground 15 meters from a hot air balloon, which starts on the ground. The balloon starts floating straight up at 20 meters per minute. Sam is in the balloon basket watching Fran on the ground. Consider the angle θ, as shown in the diagram. At what rate is θ changing, when the balloon is 25 meters above the ground?

• Diagram (Given)



• Variables

Let x = distance balloon travelled up from ground

z = distance between Sam and Fran

Given $\frac{dx}{dt} = +20$ m/min.

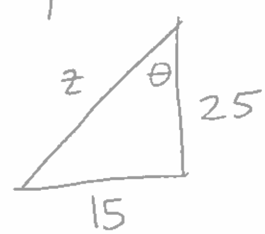
$\frac{d\theta}{dt} = ?$ when $x = 25$ m.

$$\begin{array}{r} 15 \\ \times\ 15 \\ \hline 75 \\ 150 \\ \hline 225 \end{array} \qquad \begin{array}{r} 25 \\ \times\ 25 \\ \hline 125 \\ 500 \\ \hline 625 \end{array}$$

• Equation

$$\tan\theta = \frac{15}{x}$$

• Key Moment



$$z = \sqrt{(15)^2 + (25)^2}$$
$$= \sqrt{225 + 625}$$
$$= \sqrt{850}$$

$$\Rightarrow \sec\theta = \frac{H}{A} = \frac{\sqrt{850}}{25}$$

• Differentiate

$$\frac{d}{dt}\left[\tan\theta\right] = \frac{d}{dt}\left[\frac{15}{x}\right]$$

$$\sec^2\theta \, \frac{d\theta}{dt} = \frac{-15}{x^2}\frac{dx}{dt} \quad \text{Related Rates}$$

• Substitute

$$\left(\frac{\sqrt{850}}{25}\right)^2 \cdot \frac{d\theta}{dt} = \frac{-15}{(25)^2}(20)$$

• Solve

$$\frac{d\theta}{dt} = \frac{-300}{\cancel{(25)^2}} \cdot \frac{\cancel{(25)^2}}{850} = \frac{-300}{850} = \frac{-30}{85} = \boxed{\frac{-6}{17}}$$

• Answer The angle θ is decreasing $\frac{6}{17}$ Radians per minute at this moment.

7. [14 Points]

(a) At what **Point** on the curve $y = [(x-4)\cdot \ln(x-4)] - 4x + 16$ is the tangent line horizontal?

$$y' = \cancel{(x-4)}\left[\frac{1}{\cancel{x-4}}\right] + \ln(x-4)\cdot(1) - 4$$

$$= 1 + \ln(x-4) - 4 = \ln(x-4) - 3 \overset{\text{set}}{=} 0$$

Solve $\ln(x-4) = 3$

$$\Rightarrow \cancel{e}^{\cancel{\ln}(x-4)} = e^3 \Rightarrow x - 4 = e^3 \Rightarrow x = e^3 + 4$$

y-value: $y(e^3+4) = \left[(e^3+\cancel{4}-\cancel{4})\ln(e^3+\cancel{4}-\cancel{4}) - 4(e^3+4) + 16\right]$

$$= e^3\,\cancel{\ln}(e^{3}) - 4e^3 - \cancel{16} + \cancel{16} = 3e^3 - 4e^3 = \boxed{-e^3}$$

POINT: $(x,y) = \boxed{(e^3+4,\ -e^3)}$

(b) Find the Absolute Maximum and/or Minimum **Values** for the function $f(x) = \dfrac{x^4}{e^x}$.

$$f'(x) = \frac{e^x(4x^3) - x^4\cdot e^x}{(e^x)^2} = \frac{\cancel{e^x}\cdot x^3(4-x)}{(e^x)^{\cancel{2}}} = \frac{x^3(4-x)}{\oplus\, e^x} \overset{\text{set}}{=} 0$$

$$\Rightarrow x^3(4-x) = 0 \Rightarrow x = 0 \text{ or } x = 4$$

CRITICAL NUMBERS

Sign Testing into First Derivative

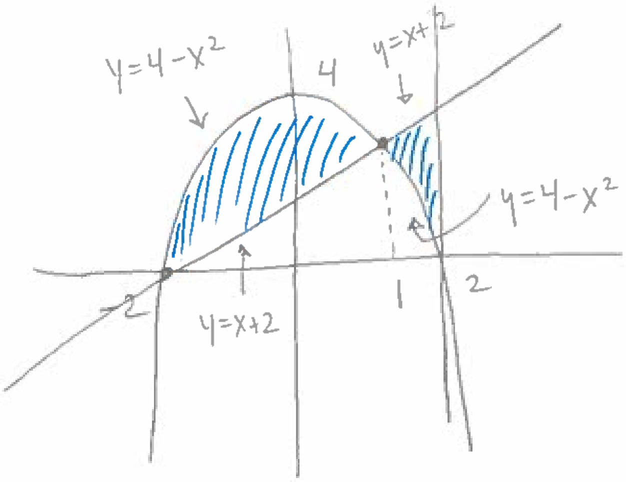
	-1	0	2	4	10
f'	$\ominus$		$\oplus$		$\ominus$
f	$\searrow$	Absolute Min	$\nearrow$	Absolute Max	$\searrow$

Absolute Max Value: $f(4) = \dfrac{4^4}{e^4} = \boxed{\dfrac{256}{e^4}}$

Absolute Min Value: $f(0) = \boxed{0}$

8. [15 Points] For each of parts (a), (b), (c), **Set-Up** but **DO NOT EVALUATE** the integral to compute the **AREA** bounded in the described regions. Sketch the region.

(a) The Region bounded by $y = 4 - x^2$, $y = x + 2$, between $x = -2$, and $x = 2$.



Intersect?

$4 - x^2 = x + 2$

$x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2$ or $x = 1$

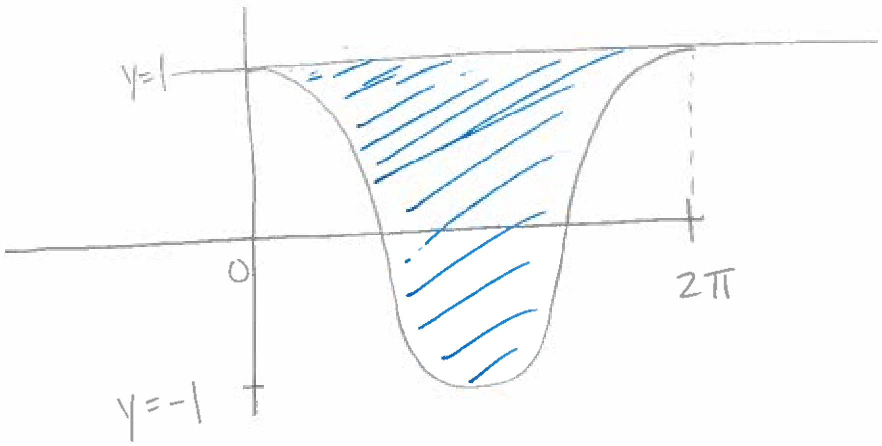
$$\text{Area} = \int_{-2}^{1} \text{Top} - \text{Bottom}\, dx + \int_{1}^{2} \text{Top} - \text{Bottom}\, dx$$

$$= \boxed{\int_{-2}^{1} 4 - x^2 - (x+2)\, dx + \int_{1}^{2} x + 2 - (4 - x^2)\, dx}$$

STOP HERE

8. (Continued) For each of parts (a), (b), (c), **Set-Up** but **DO NOT EVALUATE** the integral to compute the **AREA** bounded in the described regions. Sketch the region.

(b) The Region bounded by $y = \cos x$, $y = 1$, between $x = 0$, and $x = 2\pi$.

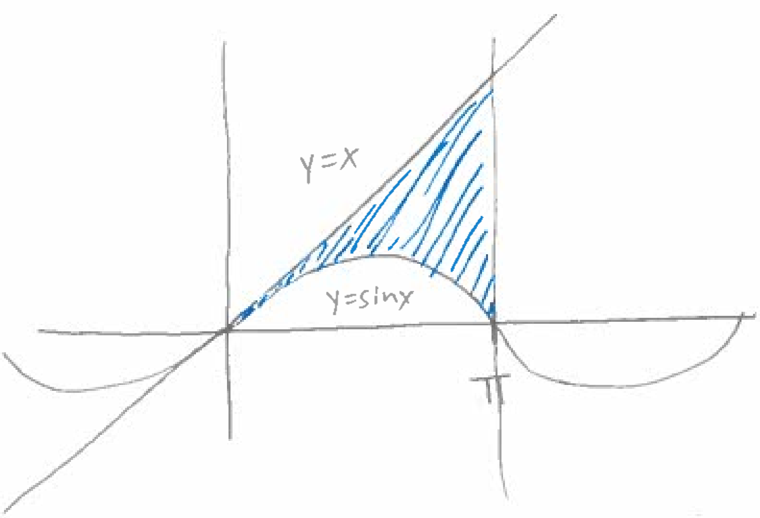


$$\text{Area} = \int_0^{2\pi} \text{Top} - \text{Bottom}\, dx = \boxed{\int_0^{2\pi} 1 - \cos x \, dx}$$

STOP

(c) The Region bounded by $y = \sin x$, $y = x$, between $x = 0$, and $x = \pi$.

(Hint: what is the tangent line to $y = \sin x$ at $x = 0$?)



Tangent Line $y=\sin x$

$y(0)=0$

$y'=\cos x$

$y'(0)=\cos 0=1$

$y-0=1(x-0)$

$y=x$

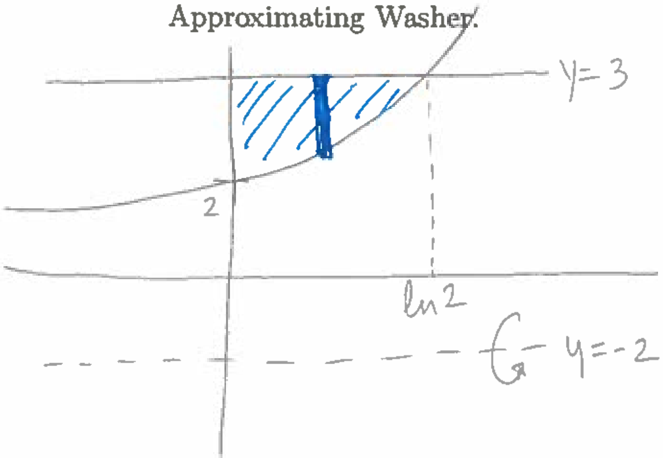
$$\text{Area} = \int_0^{\pi} \text{Top} - \text{Bottom}\, dx = \boxed{\int_0^{\pi} x - \sin x \, dx}$$

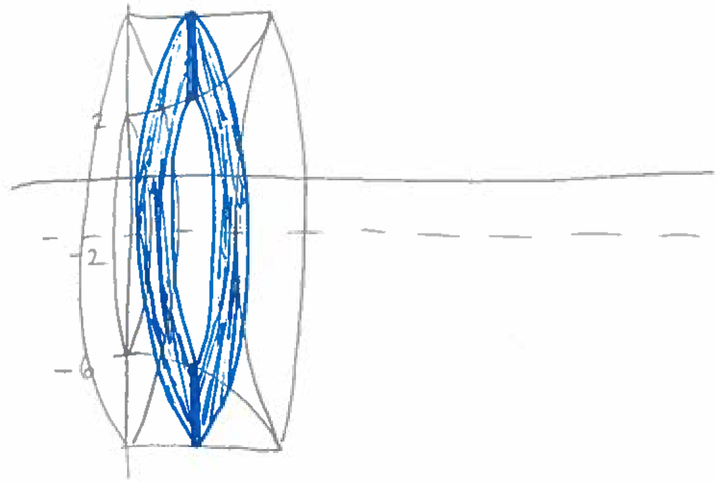
STOP

Intersect? $e^x+1=3 \Rightarrow e^x=2 \Rightarrow x=\ln 2$

9. [15 Points] Volumes

(a) Consider the region bounded by $y = e^x + 1$, $y = 3$, and $x = 0$. Sketch the bounded region. **COMPUTE** the integral to compute the Volume of the three-dimensional solid obtained by rotating the region about the horizontal line $y = -2$. Sketch the Solid of revolution, along with one Approximating Washer.





don't drop

COMPUTE

$$V = \pi \int_0^{\ln 2} (\text{Outer Radius})^2 - (\text{Inner Radius})^2 \, dx = \pi \int_0^{\ln 2} 5^2 - (e^x+3)^2 \, dx$$

$(e^{2x}+6e^x+9)$

$$= \pi \int_0^{\ln 2} 25 - e^{2x} - 6e^x - 9 \, dx = \pi \int_0^{\ln 2} 16 - e^{2x} - 6e^x \, dx$$

$$= \pi \left[16x - \frac{e^{2x}}{2} - 6e^x \right]_0^{\ln 2} = \pi \left[16\ln 2 - \frac{e^{2\ln 2}}{2} - 6e^{\ln 2} - \left(0 - \frac{e^0}{2} - 6e^0\right) \right]$$

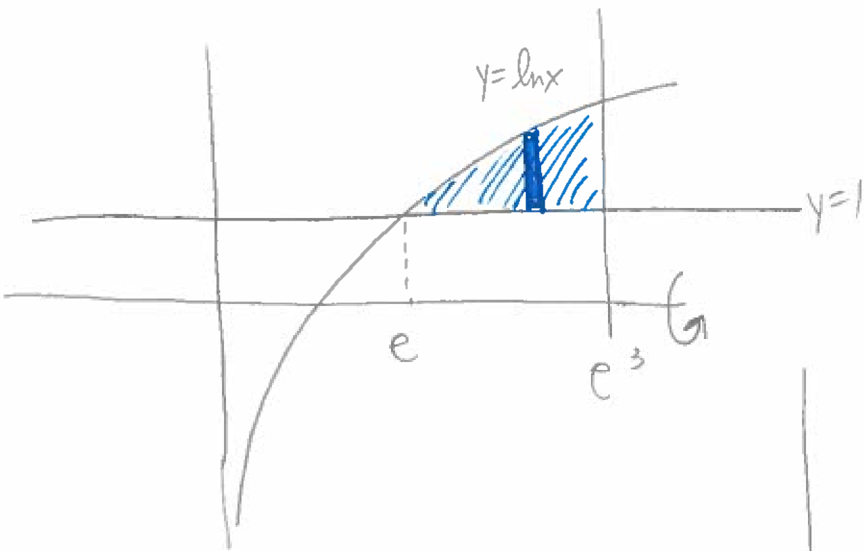
$$= \pi \left[16\ln 2 - \underbrace{\frac{4}{2}}_{-2} - 12 + \frac{1}{2} + 6 \right] = \pi \left[16\ln 2 - 8 + \frac{1}{2} \right]$$

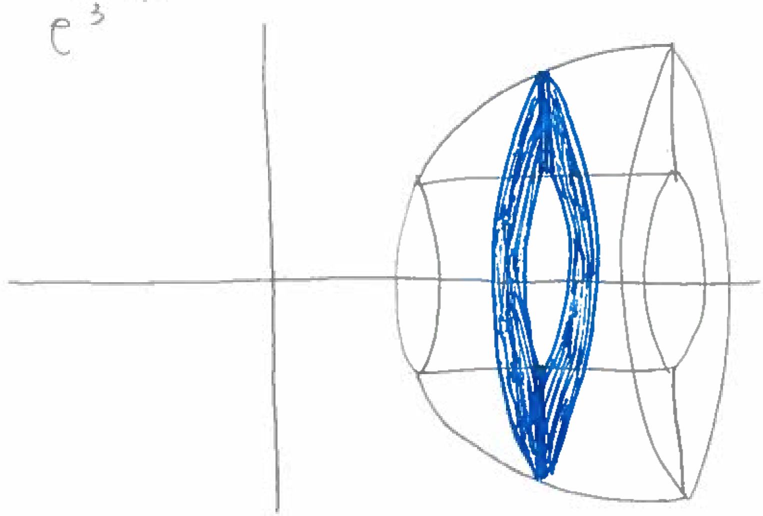
$$= \boxed{\pi \left[16\ln 2 - \frac{15}{2} \right]}$$

Intersect $\ln x = 1 \Rightarrow x = e$

9. (Continued)

(b) Consider the region bounded by $y = \ln x$, $y = 1$, and $x = e^3$. Sketch the bounded region. **Set-up** but **DO NOT EVALUATE** the integral to compute the Volume of the three-dimensional solid obtained by rotating the region about the x-axis. Sketch the Solid of revolution, along with one Approximating Washer.





$$\text{Volume} = \pi \int_e^{e^3} (\text{Outer Radius})^2 - (\text{Inner Radius})^2 \, dx$$

$$= \boxed{\pi \int_e^{e^3} (\ln x)^2 - (1)^2 \, dx}$$

STOP

10. [10 Points] Jack throws a baseball straight downward from the top of a building. This initial *speed* of the ball is 25 feet per second. The ball hits the ground with a *speed* of 153 feet per second. How tall is the building? How far above the ground is the ball one second after Jack throws it down?

(Hint: use acceleration $a(t) = -32$ feet per second squared.)

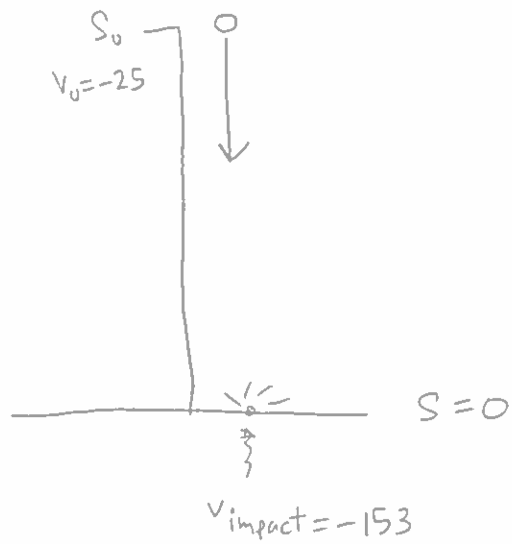
$S_0 = S(0)$ initial height

$a(t) = -32$

$v(t) = -32t + \overset{-25}{\cancel{v(0)}}$

$s(t) = -16t^2 + v(0)t + \overset{\text{unknown?}}{S_0}$

$= -16t^2 - 25t + S_0$



First $V_{impact} = -32t - 25 \overset{set}{=} -153$

$-32t = -128$

$t_{impact} = 4$ seconds

Second $s(4) = 0$

$s(4) = -16(4)^2 - 25(4) + S_0 \overset{set}{=} 0$

$= -256 - 100 + S_0 = 0$

$$\begin{array}{r} 16 \\ 16 \\ \hline 96 \\ 160 \\ \hline 256 \end{array}$$

Answer: $S_0 = \boxed{356 \text{ feet.}}$

The building was 356 feet tall. $\Rightarrow s(t) = -16t^2 - 25t + 356$

Finally, $s(1) = -16 - 25 + 356 = -41 + 356 = \boxed{315 \text{ feet.}}$

Answer: After one second has passed, the ball was 315 feet above the ground.

11. [10 Points] A population of bacteria was growing exponentially. Initially there were 3 cells. After 2 hours there were 9 cells. How many cells were there after 6 hours? When were there 123 cells?

Let $P(t) =$ population of bacteria at time t.

Solution: $P(t) = \overset{3}{\cancel{P(0)}} e^{kt}$ Given $P(0) = 3$

$P(2) = 9$

$P(t) = 3e^{kt}$ $P(6) = ?$

$P(2) = 3e^{k \cdot 2} \overset{\text{set}}{=} 9$ $P(?) = 123$

$\Rightarrow e^{2k} = 3 \Rightarrow 2k = \ln 3 \Rightarrow k = \frac{\ln 3}{2}$

Solution:

$$P(t) = 3e^{\left(\frac{\ln 3}{2}\right) \cdot t} = 3e^{\frac{t}{2}\ln 3} = 3 \cdot \cancel{e}^{\cancel{\ln}\left[(3)^{t/2}\right]} = 3 \cdot 3^{t/2}$$

Finally, $P(6) = 3 \cdot 3^{6/2} = 3 \cdot 3^3 = 3 \cdot (27) = \boxed{81 \text{ cells}}$

$P(t) = 3 \cdot 3^{t/2} \overset{\text{set}}{=} 123$

Solve $3^{t/2} = 41 \Rightarrow \ln\left[3^{t/2}\right] = \ln(41) \Rightarrow \frac{t}{2}\ln 3 = \ln(41)$

$\Rightarrow t = \boxed{\frac{2\ln(41)}{\ln 3}}$ hours (no calculator)

Answer: After Six seconds, there were 81 cells.

There were 123 cells when $\frac{2\ln(41)}{\ln 3}$ hours have passed.