Name: Answer Key

Amherst College

DEPARTMENT OF MATHEMATICS

Math 106 Final Examination

May 8, 2017

- This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.
- You need *not* simplify algebraically complicated answers. However, numerical answers such as $\sin\left(\frac{\pi}{6}\right)$, $4^{\frac{3}{2}}$, $e^{\ln 4}$, $\ln(e^7)$, $e^{-\ln 5}$, or $e^{3\ln 3}$ should be simplified.
- Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		24
2		15
3		56
4		20
5		15
6		10
7		10
8		20
9		10
10		10
11		10
Total		200

1. [24 Points] Compute each of the following derivatives.

(a) $g'(x)$, where $g(x) = (\ln x)^{\ln x}$

$$y = (\ln x)^{\ln x}$$

$$\ln y = \ln\left[(\ln x)^{\ln x}\right] = \ln x \cdot \ln[\ln x]$$

Differentiate both sides

$$\frac{d}{dx}\left[\ln y\right] = \frac{d}{dx}\left[\ln x \cdot \ln(\ln x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \cancel{\ln x} \cdot \frac{1}{\cancel{\ln x}} \cdot \left(\frac{1}{x}\right) + \ln(\ln x) \cdot \frac{1}{x} = \frac{1}{x} + \frac{\ln(\ln x)}{x}$$

Solve $$\frac{dy}{dx} = \cancel{y}\,(\ln x)^{\ln x} \cdot \left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right] = \boxed{(\ln x)^{\ln x}\left[\frac{1}{x} + \frac{\ln(\ln x)}{x}\right]}$$

(b) $\dfrac{d}{dx} \ln \left(\dfrac{3\sqrt{1+\sec^2 x}}{(8-x^3)^5\, e^{-\sin x}} \right)$ Do not simplify the final answer here.

$$= \frac{d}{dx}\left[\ln\left(3\sqrt{1+\sec^2 x}\right) - \ln\left((8-x^3)^5 \cdot e^{-\sin x}\right)\right]$$

Log Algebraic Properties

$$= \frac{d}{dx}\left[\ln 3 + \ln\sqrt{1+\sec^2 x} - \left(\ln(8-x^3)^5 + \ln e^{-\sin x}\right)\right]$$

$$= \frac{d}{dx}\left[\underbrace{\ln 3}_{\text{constant}} + \frac{1}{2}\ln(1+\sec^2 x) - 5\ln(8-x^3) \mp \sin x\right]$$

$$= \boxed{0 + \frac{1}{2}\left(\frac{1}{1+\sec^2 x}\right) \cdot 2\sec x(\sec x \tan x) - 5 \cdot \left(\frac{1}{8-x^3}\right)(-3x^2) + \cos x}$$

1. (Continued) Compute each of the following derivatives.

(c) $\frac{dy}{dx}$, if $\tan(xy) + \cancel{\ln}(e^{-9}) = \sin^2 x + (\ln(e+5))x - y^3$

-9

constant

$$\frac{d}{dx}\left[\tan(xy) - 9\right] = \frac{d}{dx}\left[\sin^2 x + (\ln(e+5))x - y^3\right]$$

$$\sec^2(xy)\left[x\cdot\frac{dy}{dx} + y(1)\right] + 0 = 2\sin x\cdot\cos x + \ln(e+5) - 3y^2\frac{dy}{dx}$$

$$x\sec^2(xy)\frac{dy}{dx} + y\sec^2(xy) = 2\sin x\cos x + \ln(e+5) - 3y^2\frac{dy}{dx}$$

Isolate.

$$x\sec^2(xy)\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 2\sin x\cos x + \ln(e+5) - y\sec^2(xy)$$

Factor

$$\left(x\sec^2(xy) + 3y^2\right)\frac{dy}{dx} = 2\sin x\cos x + \ln(e+5) - y\sec^2(xy)$$

Solve

$$\frac{dy}{dx} = \boxed{\frac{2\sin x\cos x + \ln(e+5) - y\sec^2(xy)}{x\sec^2(xy) + 3y^2}}$$

or use $\frac{1}{2}\ln x$

(d) $f'(e)$, where $f(x) = \sqrt{\ln x} - \ln\sqrt{x}$. Simplify.

$$f'(x) = \frac{1}{2\sqrt{\ln x}}\cdot\frac{1}{x} - \frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{2x\sqrt{\ln x}} - \frac{1}{2x}$$

$$f'(e) = \frac{1}{2e\sqrt{\cancel{\ln e}}} - \frac{1}{2e} = \frac{1}{2e} - \frac{1}{2e} = \boxed{0}$$

1 1

2. [15 Points] Compute each of the following derivatives. Simplify.

(a) $f'\left(\frac{\pi}{12}\right)$ where $f(x) = 2\sin^3(4x) + \sec(4x) - 8\sin(2x)$

$$f'(x) = 6\sin^2(4x)\cdot\cos(4x)\cdot 4 + \sec(4x)\tan(4x)\cdot 4 - 8\cos(2x)(2)$$

$$= 24\sin^2(4x)\cos(4x) + 4\sec(4x)\tan(4x) - 16\cos(2x)$$

$$f'(\pi/12) = 24\left[\overset{\left(\frac{\sqrt{3}}{2}\right)^2}{\cancel{\sin(\pi/3)}}\right]^2\cdot \overset{1/2}{\cancel{\cos(\pi/3)}} + 4\overset{2}{\cancel{\sec(\pi/3)}}\,\overset{\sqrt{3}}{\cancel{\tan(\pi/3)}} - 16\,\overset{\frac{\sqrt{3}}{2}}{\cancel{\cos(\pi/6)}}$$

$$= 24\left(\frac{3}{4}\right)\cdot\frac{1}{2} + 4\cdot 2\cdot\sqrt{3} - 16\left(\frac{\sqrt{3}}{2}\right)$$

$$= 9 + \cancel{8\sqrt{3}} - \cancel{8\sqrt{3}} = \boxed{9}$$

(b) $f'\left(\frac{\pi}{4}\right)$ where $f(x) = \cos(2x) + \overset{(\tan x)^{-2}}{\frac{1}{\tan^2 x}} + \sin\left(x - \frac{\pi}{4}\right)$

$$f'(x) = -\sin(2x)\,2 - 2(\tan x)^{-3}\cdot\sec^2 x + \cos(x - \pi/4)\,(1)$$

$$f'(\pi/4) = -2\overset{1}{\cancel{\sin(\pi/2)}} - \frac{2}{(\underset{1}{\cancel{\tan \pi/4}})^3}\cdot \underset{(\sqrt{2})^2}{\cancel{\sec^2(\pi/4)}} + \overset{1}{\cancel{\cos 0}}$$

$$= -2 - 2\cdot 2 + 1 = -2 - 4 + 1 = \boxed{-5}$$

3. [56 Points] Compute each of the following integrals. SPLIT now

(a) $\displaystyle\int \frac{\left(1-x^{\frac{3}{4}}\right)\left(x^{\frac{5}{4}}-x^{3}\right)}{x^{3}}\,dx = \int \frac{x^{5/4}-x^{3}-\cancel{x^{8/4}}^{\,x^2}+x^{3/4}x^{3}}{x^{3}}\,dx$

$$= \int \frac{x^{5/4}}{\underset{x^{12/4}}{x^{3}}} - \frac{\cancel{x^{3}}}{\cancel{x^{3}}} - \frac{x^{2}}{x^{3}} + \frac{x^{3/4}\cdot \cancel{x^{3}}}{\cancel{x^{3}}}\,dx = \int x^{-7/4} - 1 - \frac{1}{x} + x^{3/4}\,dx$$

$$= \boxed{\frac{-4}{3}x^{-3/4} - x - \ln|x| + \frac{4}{7}x^{7/4} + C}$$

(b) Show that $\displaystyle\int_{-1}^{2} \frac{\overbrace{x^{3}}^{x^2\cdot x}}{x^{2}-5}\,dx = \frac{3}{2} - \ln(32)$

$x^2 = u+5$

$$\boxed{\begin{aligned} u &= x^2-5 \\ du &= 2x\,dx \\ \tfrac{1}{2}du &= x\,dx \end{aligned}}$$

$$\boxed{\begin{aligned} x=-1 &\Rightarrow u=(-1)^2-5=-4 \\ x=2 &\Rightarrow u=4-5=-1 \end{aligned}}$$

$$= \frac{1}{2}\int_{-4}^{-1} \frac{u+5}{u}\,du = \frac{1}{2}\int_{-4}^{-1} 1+\frac{5}{u}\,du$$

SPLIT

$$= \frac{1}{2}\Big[u+5\ln|u|\Big]\Big|_{-4}^{-1} = \frac{1}{2}\Big[-1+5\cancel{\ln|-1|}^{\,\ln 1 \to 0} - (-4+5\ln 4)\Big]$$

$$= \frac{1}{2}\Big[\overbrace{-1+4}^{3} - 5\ln 4\Big] = \frac{3}{2} - \frac{5}{2}\ln 4 = \frac{3}{2} - \ln\Big[\underbrace{(4)^{5/2}}_{32}\Big]$$

$$= \boxed{\frac{3}{2} - \ln(32)}$$

Match!

3. (Continued) Compute each of the following integrals.

(c) $\int \frac{(1+e^{3x})^2}{e^{3x}}\,dx \overset{\text{FOIL}}{=} \int \frac{1+2e^{3x}+e^{6x}}{e^{3x}}\,dx \overset{\text{SPLIT}}{=} \int \frac{1}{e^{3x}} + \frac{2\cancel{e^{3x}}}{\cancel{e^{3x}}} + \frac{e^{6x}}{e^{3x}}\,dx$

$= \int e^{-3x} + 2 + e^{3x}\,dx \overset{\text{k-rule}}{=} \frac{e^{-3x}}{-3} + 2x + \frac{e^{3x}}{3} + C$

$= \boxed{\frac{-1}{3e^{3x}} + 2x + \frac{e^{3x}}{3} + C}$

(d) $\int \frac{e^{3x}}{(1+e^{3x})^2}\,dx = \frac{1}{3}\int \frac{1}{u^2}\,du = \frac{1}{3}\left(\frac{u^{-1}}{-1}\right) + C$

u^{-2}

$\boxed{\begin{array}{l} u = 1+e^{3x} \\ du = 3e^{3x}dx \\ \frac{1}{3}du = e^{3x}dx \end{array}}$

$= \frac{-1}{3u} + C$

$= \boxed{\frac{-1}{3(1+e^{3x})} + C}$

3. (Continued) Compute each of the following integrals.

(e) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \tan^3 x \, dx = \int_{1/\sqrt{3}}^{\sqrt{3}} u^3 du = \left. \frac{u^4}{4} \right|_{1/\sqrt{3}}^{\sqrt{3}}$

$u = \tan x$

$du = \sec^2 x \, dx$

$= \frac{1}{4}\left[(\sqrt{3})^4 - \frac{1}{(\sqrt{3})^4} \right]$

$x = \pi/6 \Rightarrow u = \tan(\pi/6) = \frac{1}{\sqrt{3}}$

$x = \frac{\pi}{3} \Rightarrow u = \tan(\pi/3) = \sqrt{3}$

$= \frac{1}{4}\left[9 - \frac{1}{9} \right] = \frac{1}{4}\left[\frac{80}{9} \right] = \boxed{\frac{20}{9}}$

(f) $\int \frac{1}{5\sqrt{x}\, e^{\sqrt{x}}} dx = \frac{2}{5}\int \frac{1}{e^u} du = \frac{2}{5}\int e^{-u} du = \frac{2}{5}\left(\frac{e^{-u}}{(-1)} \right) + C$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}} dx$

$2du = \frac{1}{\sqrt{x}} dx$

$= \frac{-2}{5e^u} + C = \boxed{\frac{-2}{5e^{\sqrt{x}}} + C}$

3. (Continued) Compute each of the following integrals.

(g) $\int_2^6 \frac{1}{x^2}\cos\left(\frac{\pi}{x}\right)\,dx = -\frac{1}{\pi}\int_{\pi/2}^{\pi/6} \cos u\,du = -\frac{1}{\pi}\cdot \sin u\Big|_{\pi/2}^{\pi/6}$

$u = \pi/x$

$du = -\frac{\pi}{x^2}\,dx$

$-\frac{1}{\pi}\,du = \frac{1}{x^2}\,dx$

$x = 2 \Rightarrow u = \pi/2$

$x = 6 \Rightarrow u = \pi/6$

$= -\frac{1}{\pi}\left[\overbrace{\cancel{\sin}(\pi/6)}^{1/2} - \overbrace{\cancel{\sin}(\pi/2)}^{1}\right]$

$= -\frac{1}{\pi}\left[\frac{1}{2} - 1\right] = -\frac{1}{\pi}\left[-\frac{1}{2}\right] = \boxed{\frac{1}{2\pi}}$

(h) $\int_1^{e^3} \frac{\sqrt{1+\ln x}}{x}\,dx = \int_1^4 \overbrace{\sqrt{u}}^{u^{1/2}}\,du = \frac{2}{3}u^{3/2}\Big|_1^4$

$u = 1 + \ln x$

$du = \frac{1}{x}\,dx$

$x = 1 \Rightarrow u = 1 + \overbrace{\cancel{\ln 1}}^{0} = 1$

$x = e^3 \Rightarrow u = 1 + \overbrace{\cancel{\ln e^3}}^{3} = 4$

$= \frac{2}{3}\left[\overbrace{\cancel{4^{3/2}}}^{\cancel{2^3}\ 8} - \overbrace{\cancel{1^{3/2}}}^{1}\right] = \frac{2}{3}\left[\overbrace{8-1}^{7}\right] = \boxed{\frac{14}{3}}$

4. [20 Points] Consider the function given by

$$e^{-7x} \qquad 7e^{-x} \qquad ex^{-7} \qquad x^{-e} \qquad ex^{-1} \qquad \tfrac{1}{e}x^{-1}$$

$$f(x) = e^{7x} + \frac{1}{e^{7x}} + e^7 + \frac{7}{e^x} + \frac{7}{e^7} + \frac{e}{x^7} + x^e + \frac{1}{x^e} + \frac{x}{e} + \frac{e}{x} + ex + \frac{1}{ex}$$

constant.

(a) Compute the **derivative**, $f'(x)$.

$$f'(x) = \boxed{7e^{7x} - 7e^{-7x} + 0 - 7e^{-x} + 0 - 7ex^{-8} + ex^{e-1} - ex^{-e-1} + \frac{1}{e} - ex^{-2} + e - \frac{1}{e}x^{-2}}$$

(b) Compute the **antiderivative**, $\int f(x)\,dx$.

$$\int f(x)dx = \boxed{\frac{e^{7x}}{7} + \frac{e^{-7x}}{(-7)} + e^7 \cdot x + \frac{7e^{-x}}{(-1)} + \frac{7}{e^7}\cdot x + \frac{ex^{-6}}{(-6)} + \frac{x^{e+1}}{e+1} + \frac{x^{-e+1}}{-e+1} + \frac{x^2}{2e} + e\ln|x| \cdots + \frac{ex^2}{2} + \frac{1}{e}\ln|x| + C}$$

5. [15 Points] Compute $\int_{-1}^{1} x^3\,dx$ using each of the following two different methods:

(*a*) Fundamental Theorem of Calculus,
(*b*) The limit definition of the definite integral.

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$

$(-1 + \frac{2i}{n})^3 = (-1)^3 + 3(-1)^2(\frac{2i}{n}) + 3(-1)(\frac{2i}{n})^2 + (\frac{2i}{n})^3$

$= -1 + 6i/n - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}$

Recall

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2 \quad\text{and}\quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad\text{and}\quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad\text{and}\quad \sum_{i=1}^{n} 1 = n$$

a. $\int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = \frac{1^4}{4} - \frac{(-1)^4}{4} = \frac{1}{4} - \frac{1}{4} = \boxed{0}$

b. $f(x) = x^3$

$a = -1,\ b = 1$

$\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{n} = \frac{2}{n}$

$x_i = a + i\Delta x$

$= -1 + \frac{2i}{n}$

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(-1 + 2i/n) \cdot \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(-1 + \frac{2i}{n}\right)^3 = \lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} -1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}$$

expand

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^{n} -1 + \frac{2}{n}\sum_{i=1}^{n} \frac{6i}{n} + \frac{2}{n}\sum_{i=1}^{n} \frac{-12i^2}{n^2} + \frac{2}{n}\sum_{i=1}^{n} \frac{8i^3}{n^3}$$

$$= \lim_{n\to\infty} -\frac{2}{n}\sum_{i=1}^{n} 1 + \frac{12}{n^2}\sum_{n=1}^{n} i - \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} -\frac{2}{\cancel{n}}(\cancel{n}) + \frac{12}{n^2}\left[\frac{n(n+1)}{2}\right] - \frac{24}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right] + \frac{16}{n^4}\left[\frac{n^2(n+1)^2}{2^2 \to 4}\right]$$

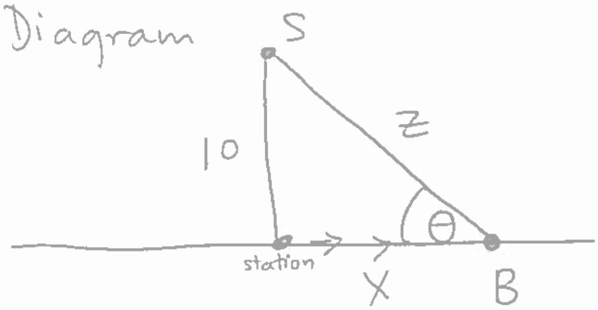
$$= \lim_{n\to\infty} -2 + 6\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right) - 4\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) + 4\left(\frac{n}{n}\right)\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{n+1}{n}\right)$$

$$= \lim_{n\to\infty} -2 + 6(1)(1 + \cancel{1/n}^{\,0}) - 4(1)(1 + \cancel{1/n}^{\,0})(2 + \cancel{1/n}^{\,0}) + 4(1)(1)(1 + \cancel{1/n}^{\,0})(1 + \cancel{1/n}^{\,0})$$

$= -2 + 6 - 8 + 4 = -10 + 10 = \boxed{0}$ Match!

6. [10 Points] Sally is at a train station, standing 10 meters from the railroad track as a train goes past. She is waiting for her friend Bob, who is on the train looking at Sally through the window. The train misses its stop. At the moment when the distance between Sally and Bob is 13 meters, Bob's head is rotating at a rate of 2 radians per second to keep her in sight. How fast is the train going at that moment?

- Diagram



- Variables

Let θ = angle Bob's head rotates from track

x = distance train has moved past stop.

z = distance between Bob and Sally.

- Given

$$\frac{d\theta}{dt} = -2 \text{ rad./sec.}$$

$$\frac{dx}{dt} = ? \text{ when } z = 13$$

- Equation

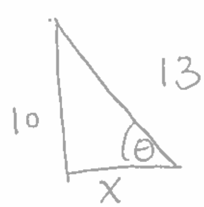
$$\tan\theta = \frac{10}{x}$$

- Differentiate

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{-10}{x^2} \cdot \frac{dx}{dt}$$

(Key Moment)

- Extra Solvable Information



$$\Rightarrow x = \sqrt{13^2 - 10^2}$$

$$x = \sqrt{69}$$

$$\Rightarrow \sec\theta = \frac{H}{A} = \frac{13}{\sqrt{69}}$$

- Substitute

$$\cancel{\left(\frac{13}{\sqrt{69}}\right)^2} \cdot (-2) = \frac{\cancel{-}10}{\cancel{(\sqrt{69})^2}} \cdot \frac{dx}{dt}$$

- Solve $169\left(\frac{\cancel{2}}{\cancel{10}_{5}}\right) = \frac{dx}{dt}$

- Answer: The train is moving $\frac{169}{5}$ meters per second

7. [10 Points]

(a) Compute $\int \tan x \, dx.$ $= \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{u} du = -\ln|u| + C$

$u = \cos x$

$du = -\sin x \, dx$

$-du = -\sin x \, dx$

$= \boxed{-\ln|\cos x| + C}$

(b) Consider $f(x)$ with $f''(x) = \sec^2 x$ and $f'\left(\frac{\pi}{3}\right) = 2\sqrt{3}$ and $f(0) = 7$. Find $f(x)$.

$f''(x) = \sec^2 x$

$f'(x) = \int \sec^2 x \, dx = \tan x + C_1$

$f'(\pi/3) = \underbrace{\tan(\pi/3)}_{\sqrt{3}} + C_1 \overset{\text{set}}{=} 2\sqrt{3} \Rightarrow C_1 = \sqrt{3}$

$f'(x) = \tan x + \sqrt{3}$

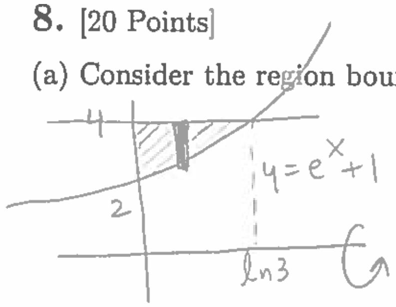
$f(x) = \int \tan x + \sqrt{3} \, dx \underset{\substack{\text{See} \\ (a).}}{=} -\ln|\cos x| + \sqrt{3}x + C_2$

$f(0) = -\underbrace{\ln|\underbrace{\cos 0}_{1}|}_{0} + 0 + C_2 \overset{\text{set}}{=} 7 \Rightarrow C_2 = 7$

Finally, $f(x) = \boxed{-\ln|\cos x| + \sqrt{3}\,x + 7}$

8. [20 Points]

(a) Consider the region bounded by $y = e^x + 1$, $y = 4$, and $x = 0$. Sketch the bounded region.



Intersect?

$e^x + 1 = 4$

$e^x = 3$

$x = \ln 3$

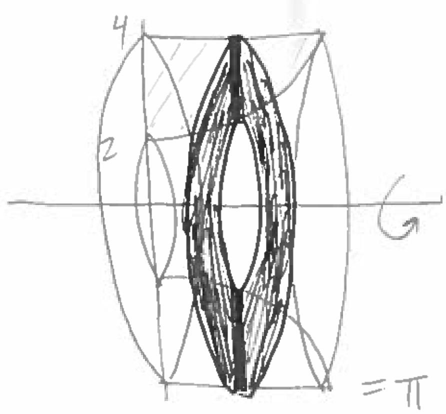
(b) Compute the area of the bounded region in (a). Simplify.

$$\text{Area} = \int_0^{\ln 3} 4-(e^x+1)\,dx = \int_0^{\ln 3} 4-e^x-1\,dx = \int_0^{\ln 3} 3-e^x\,dx$$

$$= 3x - e^x \Big|_0^{\ln 3} = 3\ln 3 - e^{\ln 3} - (0 - \cancel{e^0}^{\,1})$$

$$= 3\ln 3 - 3 + 1 = \boxed{3\ln 3 - 2}$$

(c) Compute the volume of the three-dimensional solid obtained by rotating the region in (a) about the $\boldsymbol{x}$**-axis**. Sketch the solid, along with one of the approximating washers. Simplify.



$$V = \pi \int_0^{\ln 3} (\text{Outer Radius})^2 - (\text{Inner Radius})^2\,dx$$

$$-(e^{2x}+2e^x+1)$$

$$= \pi \int_0^{\ln 3} 4^2 - (e^x+1)^2\,dx$$

$$= \pi \int_0^{\ln 3} 16 - e^{2x} - 2e^x - 1\,dx = \pi \int_0^{\ln 3} 15 - e^{2x} - 2e^x\,dx$$

$$= \pi \left[15x - \frac{e^{2x}}{2} - 2e^x\right]\Bigg|_0^{\ln 3} = \pi\left[15\ln 3 - \frac{e^{2\ln 3}}{2} - 2e^{\ln 3} - \left(0 - \cancel{\frac{e^0}{2}}^{\,1/2} - \cancel{2e^0}^{\,-2}\right)\right]$$

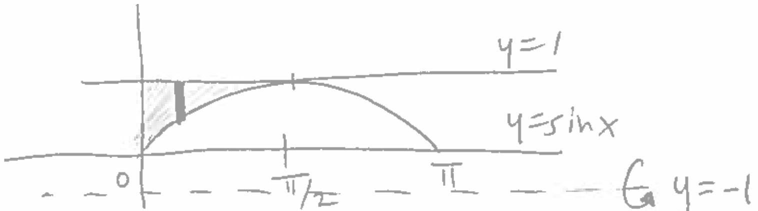
$$= \pi\left[15\ln 3 - \frac{9}{2} - 6 + \frac{1}{2} + 2\right] = \boxed{\pi\left[15\ln 3 - 8\right]}$$

-4

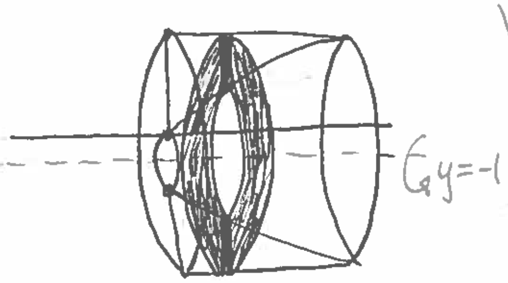
$-8/2 \to -4$

8. (Continued)

(d) Consider a **different** region bounded by $y = \sin x$, $y = 1$, $x = 0$ and $x = \frac{\pi}{2}$. Sketch the bounded region.



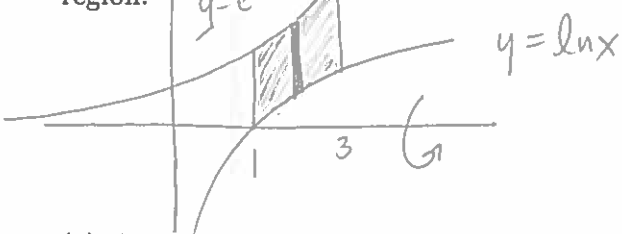
(e) **Set-Up** but **DO NOT EVALUATE** the integral to compute the volume of the three-dimensional solid obtained by rotating the region in (d) about the horizontal line $\boldsymbol{y = -1}$. Sketch the solid, along with one of the approximating washers.



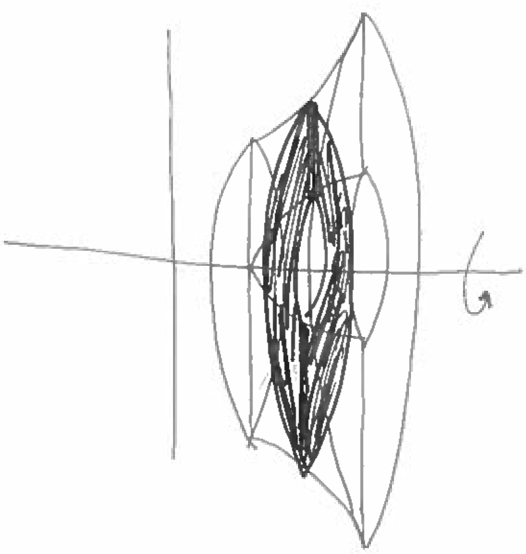
$$V = \pi \int_0^{\pi/2} (\text{Outer Radius})^2 - (\text{Inner Radius})^2 \, dx$$

$$\boxed{\pi \int_0^{\pi/2} 2^2 - (\sin x + 1)^2 \, dx}$$

(f) Consider a **different** region bounded by $y = e^x$, $y = \ln x$, $x = 1$, and $x = 3$. Sketch the bounded region.



(g) **Set-Up** but **DO NOT EVALUATE** the integral to compute the volume of the three-dimensional solid obtained by rotating the region in (f) about the $\boldsymbol{x}$**-axis**. Sketch the solid, along with one of the approximating washers.



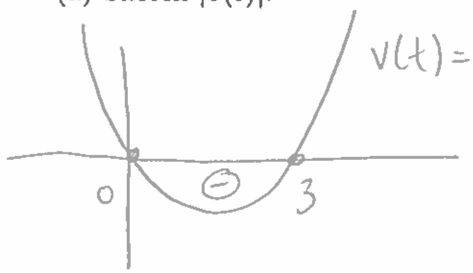
$$V = \pi \int_1^3 (\text{Outer Radius})^2 - (\text{Inner Radius})^2 \, dx$$

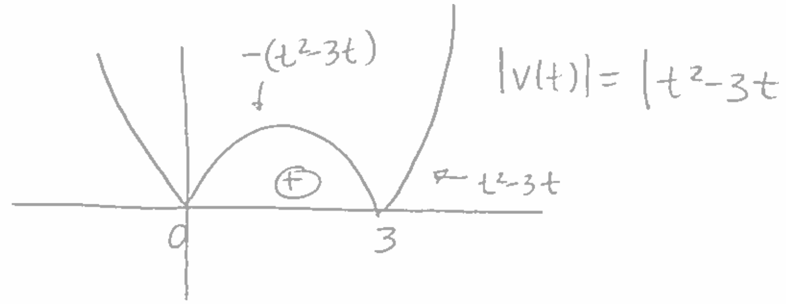
$$= \boxed{\pi \int_1^3 (e^x)^2 - (\ln x)^2 \, dx}$$

$t^2-3t = t(t-3) \overset{\text{set}}{=} 0 \Rightarrow t=0 \quad t=3.$

9. [10 Points] Consider an object moving on a number line such that its velocity at time t seconds is given by $v(t) = t^2 - 3t$ feet per second.

(a) Sketch $|v(t)|$.





(b) Compute the **total distance** travelled for $0 \le t \le 4$. Simplify.

$$\text{Total Distance} = \int_0^4 |v(t)|\,dt$$

$$= \int_0^4 |t^2-3t|\,dt = \int_0^3 -(t^2-3t)\,dt + \int_3^4 t^2-3t\,dt$$

$$= \int_0^3 -t^2+3t\,dt + \int_3^4 t^2-3t\,dt$$

$$= -\frac{t^3}{3} + \frac{3t^2}{2}\Big|_0^3 + \frac{t^3}{3} - \frac{3t^2}{2}\Big|_3^4$$

$$= -\frac{\cancel{27}^{\,9}}{\cancel{3}} + \frac{27}{2} - (0+0) + \frac{64}{3} - 24 - \left(\frac{\cancel{27}^{\,9}}{\cancel{3}} - \frac{27}{2}\right)$$

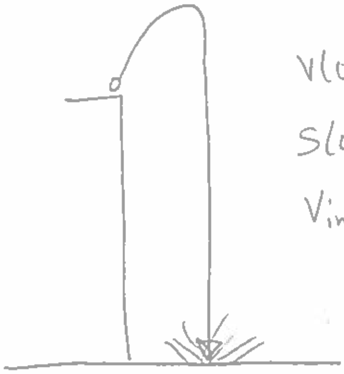
$$= -9 + \frac{27}{2} + \frac{64}{3} - 24 - 9 + \frac{27}{2}$$

$$= -42 + \frac{\cancel{54}^{\,27}}{\cancel{2}} + \frac{64}{3} = -15 + \frac{64}{3} = \frac{-45}{3} + \frac{64}{3} = \boxed{\frac{19}{3}}$$

-18
-24
-42.

-42
+27
-15

10. [10 Points] Mark throws a baseball upward from the top of a building. This initial *speed* of the ball is 20 feet per second. On its way down, the ball hits the ground with a *speed* of 44 feet per second. How tall is the building? (Hint: use acceleration $a(t) = -32$ feet per second squared.)



$v(0) = +20$ ft/sec.

$s(0) = ?$

$V_{impact} = -44$ ft/sec.

$a(t) = -32$

$v(t) = -32t + \cancel{v(0)}^{+20} = -32t + 20$

$s(t) = -16t^2 + v(0)t + s(0) = -16t^2 + 20t + s(0)$?

Time @ Impact

$v(t_{impact}) = -32t + 20 \overset{set}{=} -44 \Rightarrow -32t = -64 \Rightarrow t_{impact} = 2$ seconds

Position @ Impact

$s(2) = -16(2)^2 + 20(2) + s(0) \overset{set}{=} 0$ ← ground!

or S_0

$-64 + 40 + S_0 = 0$

$-24 + S_0 = 0 \Rightarrow S_0 = 24$ feet.

Finally,

$\boxed{\text{The height of the building was 24 feet.}}$

11. [10 Points] A population of bacteria was growing exponentially. Initially there were 4 cells. After 1 hour there were 12 cells. How many cells were there after 3 hours? When were there 324 cells?

Amount of Bacteria

$P(t) = \cancel{P(0)}^{4} e^{kt}$

Given $P(0) = 4$

$P(1) = 12$

$P(3) = ?$

$P(?) = 324$

$P(t) = 4e^{kt}$

$P(1) = 4e^{k} \overset{\text{set}}{=} 12$

Solve $e^{k} = \frac{12}{4} = 3$

$\Rightarrow k = \ln 3$

$P(t) = 4e^{(\ln 3)t} = 4\cancel{e}^{\cancel{\ln}(3^t)} = 4 \cdot 3^t$

$P(3) = 4 \cdot 3^3 = 4 \cdot (27) = 108$ cells.

Answer: $\boxed{\text{There were 108 cells after 3 hours.}}$

$P(t) = 4 \cdot 3^t \overset{\text{set}}{=} 324$

$\Rightarrow 3^t = \frac{324}{4} = 81$

$\Rightarrow t = 4$ hours.

Answer: $\boxed{\text{There were 324 cells after 4 hours.}}$