Extra Riemann Sum, Definite Integral Example

Evaluate $\int_0^3 x^2\, dx$ using Riemann Sums and the limit definition of the definite integral.

Here $f(x) = x^2, \quad a = 0, \quad b = 3, \quad \Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$
and $x_i = a + x_i = 0 + i\left(\frac{3}{n}\right) = \frac{3i}{n}$.

$$\begin{aligned}
\int_0^3 x^2\, dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \left(\frac{3i}{n}\right)^2 \\
&= \lim_{n\to\infty} \frac{3}{n} \sum_{i=1}^{n} \frac{9i^2}{n^2} \\
&= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty} \frac{27}{6}\left(\frac{n}{n}\right)\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\
&= \lim_{n\to\infty} \frac{27}{6}(1)\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \\
&= \frac{27}{6}(1)(1)(2) = \frac{27}{3} \\
&= \boxed{9}
\end{aligned}$$