Summer Academy, Calculus with Algebra, 2019 ANSWER KEY

Worksheet 1, Tuesday, June 25, 2019

1. Simplify each of the following expressions. Show your work.

   We clear the denominator by flipping and multiplying...

   (a) $$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} \cdot \frac{\left(\frac{d}{c}\right)}{\left(\frac{d}{c}\right)} = \frac{\left(\frac{a}{b}\right)\left(\frac{d}{c}\right)}{1} = \boxed{\frac{ad}{bc}}$$

   (b) $$\frac{1}{\left(\frac{a}{b}\right)} = \frac{1}{\left(\frac{a}{b}\right)} \cdot \frac{\left(\frac{b}{a}\right)}{\left(\frac{b}{a}\right)} = \frac{\left(\frac{b}{a}\right)}{1} = \boxed{\frac{b}{a}}$$

   (c) $$\frac{\left(\frac{a}{b}\right)}{c} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{1}\right)} = \frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{1}\right)} \cdot \frac{\left(\frac{1}{c}\right)}{\left(\frac{1}{c}\right)} = \frac{\left(\frac{a}{b}\right) \cdot \left(\frac{1}{c}\right)}{1} = \boxed{\frac{a}{bc}}$$

   (d) $$\frac{a}{\left(\frac{b}{c}\right)} = \frac{a}{\left(\frac{b}{c}\right)} \cdot \frac{\left(\frac{c}{b}\right)}{\left(\frac{c}{b}\right)} = \frac{a \cdot \left(\frac{c}{b}\right)}{1} = \boxed{\frac{ac}{b}}$$

2. Solve each of the following equations (if possible):

   (a) $x^2 - 4x - 21 = 0$

   Factor $(x-7)(x+3) = 0$ means either $x - 7 = 0$ or $x + 3 = 0$. Finally, $\boxed{x = 7}$ or $\boxed{x = -3}$.

   (b) $x^2 - x + 7 = 0$

   Try the Quadratic Formula:

   $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Here $a = 1$, $b = -1$ and $c = 7$.

   Then $x = \dfrac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(7)}}{2(1)} = \dfrac{1 \pm \sqrt{1 - 28}}{2} = \dfrac{1 \pm \sqrt{-27}}{2}$

   $\boxed{\text{No Real solution}}$ because we have a negative discriminant $(b^2 - 4ac)$.

   (c) $x^2 + 2x - 4 = 0$

   Again, try the Quadratic Formula:

   $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Here $a = 1$, $b = 2$ and $c = -4$.

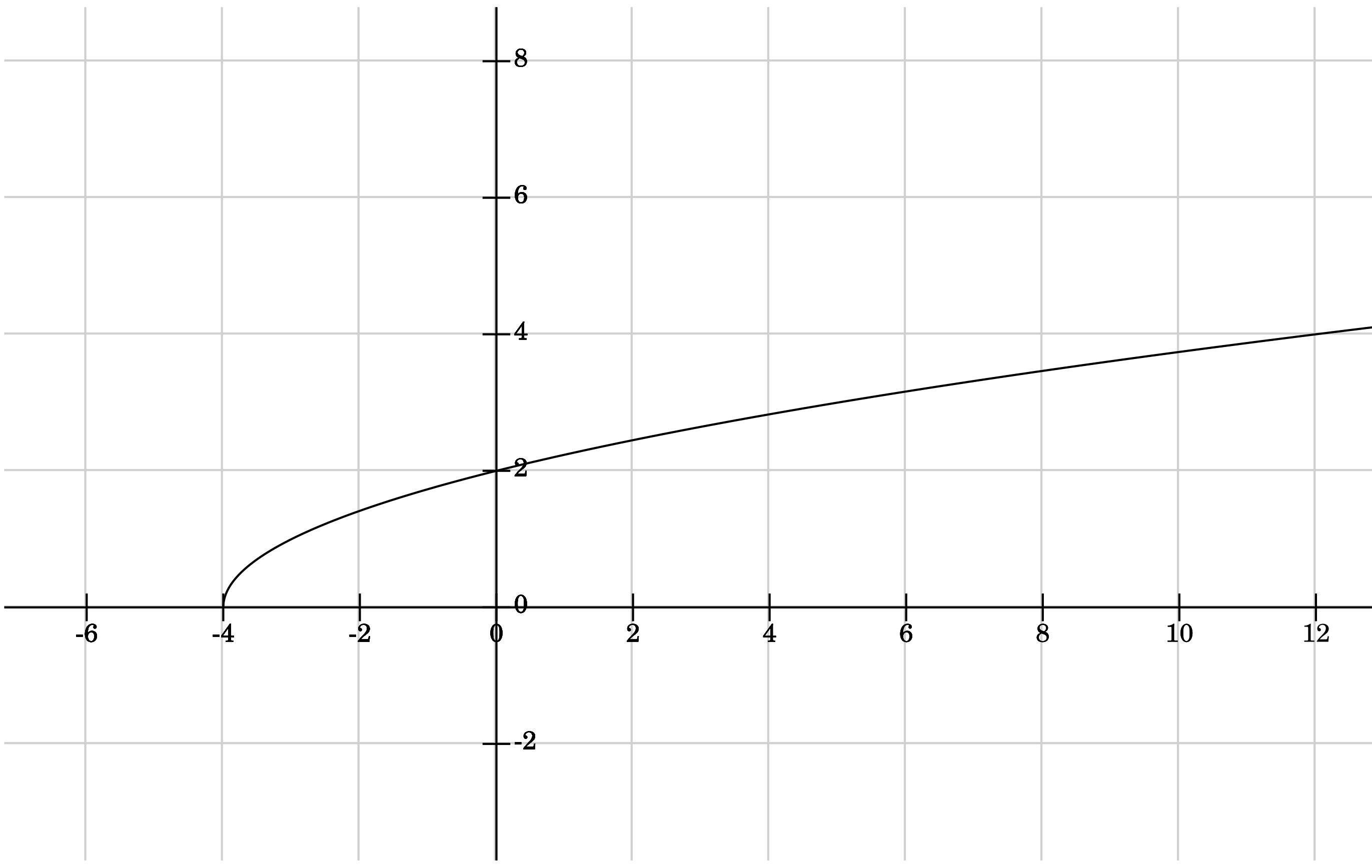
Then $x = \dfrac{-2 \pm \sqrt{2^2 - 4(1)(-4)}}{2(1)} = \dfrac{-2 \pm \sqrt{4+16}}{2} = \dfrac{-2 \pm \sqrt{20}}{2} = \dfrac{-2 \pm \sqrt{4 \cdot 5}}{2} = \dfrac{-2 \pm \sqrt{4}\sqrt{5}}{2} =$
$\dfrac{-2 \pm 2\sqrt{5}}{2} = \boxed{-1 \pm \sqrt{5}}$

3. YES or NO: Does $\sqrt{x^2+4} = x+2$? Why or why not?
   NO, equal functions must take the same value at *every* point. Here test $x = 1$. $\sqrt{5} \neq 3$.

4. Recall from class that we saw the graphs of $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{x}$. Use these graphs to help you do the following:

(a) Sketch the graph of $F(x) = \sqrt{x+4}$. Discuss the Domain and Range for this new function.

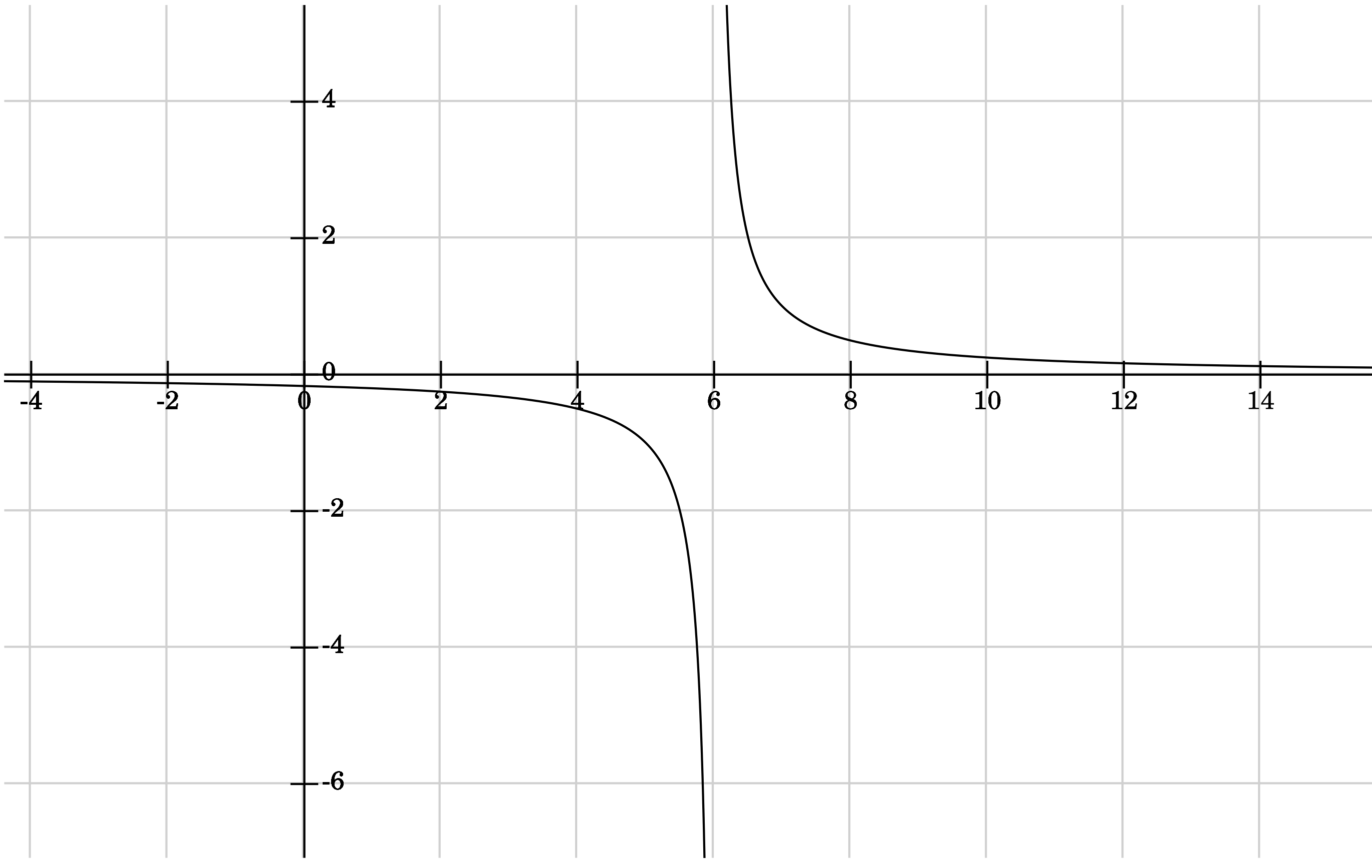
Domain=$\{x : x \geq -4\}$ Range=$[0, \infty) = \{y : y \geq 0\}$



(b) Sketch the graph of $G(x) = \dfrac{1}{x-6}$. Discuss the Domain and Range for this new function. Discuss the output behavior of $G(x)$ as the input value x is near $x = 6$. (Be specific.) Discuss the output behavior of $G(x)$ out near $\pm\infty$.

Domain=$\{x : x \neq 6\}$ Range=$(-\infty, 0) \cup (0, \infty) = \{y : y \neq 0\}$

As x approaches 6 from the positive direction (the right), then function output values are blowing up to ∞. As x approaches 6 from the negative direction (the left), then the function output values are blowing down to $-\infty$. As x approaches $+\infty$ the output values are approaching 0, from the positive direction. (0^+) As x approaches $-\infty$ the output values are approaching 0, but from the negative direction. (0^-)



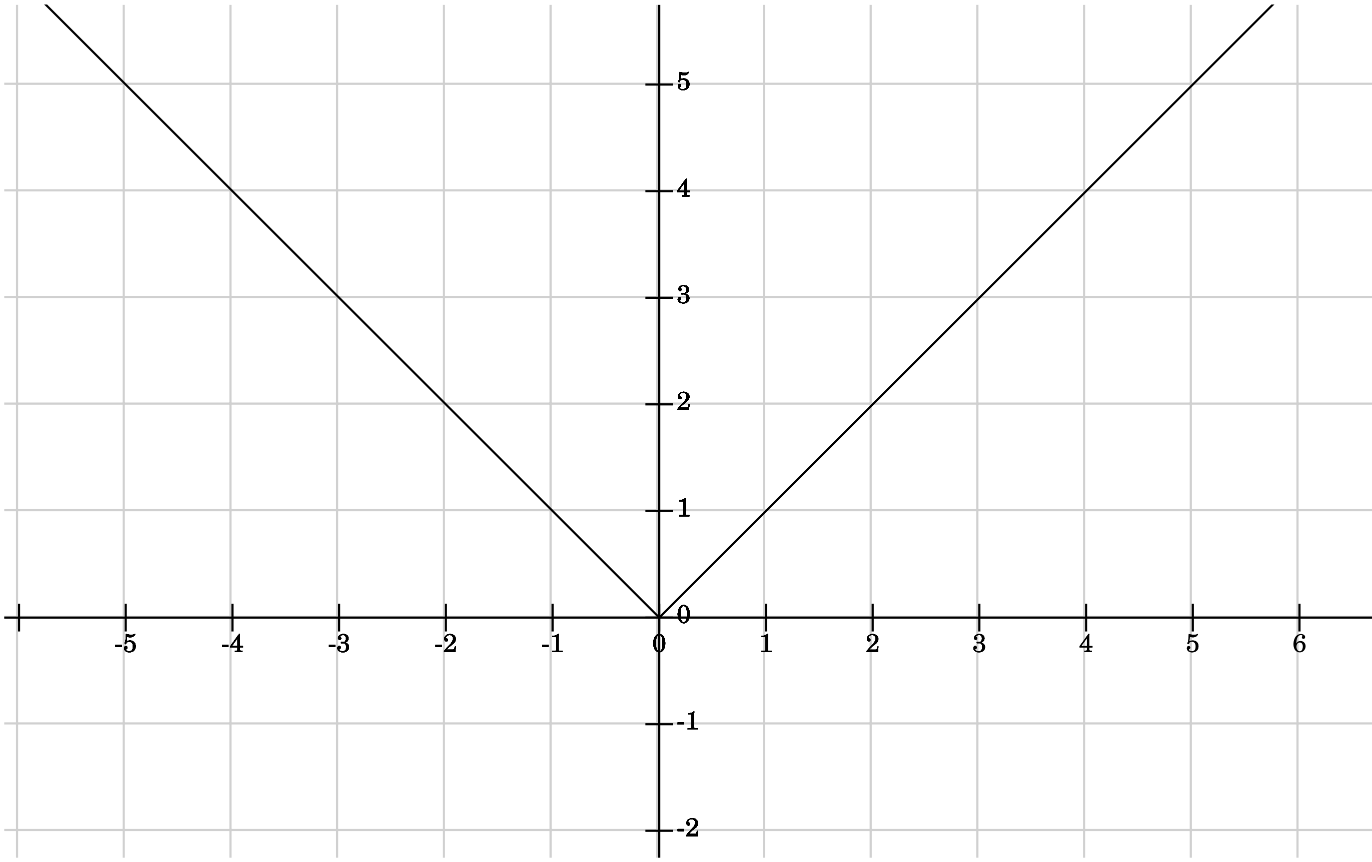
5. The Absolute Value Function $f(x) = |x|$ is a *piece-wise defined function* defined by

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

(a) Give the Domain and Range for this function. Graph the absolute value function. Discuss how this function behaves near $x = 0$.

Domain=$\mathbb{R}$ Range=$\{y : y \geq 0\}$

For $x > 0$ the graph has slope 1 and for $x < 0$ the graph has slope -1. Both pieces of the graph merge together at $x = 0$ with output 0 there.

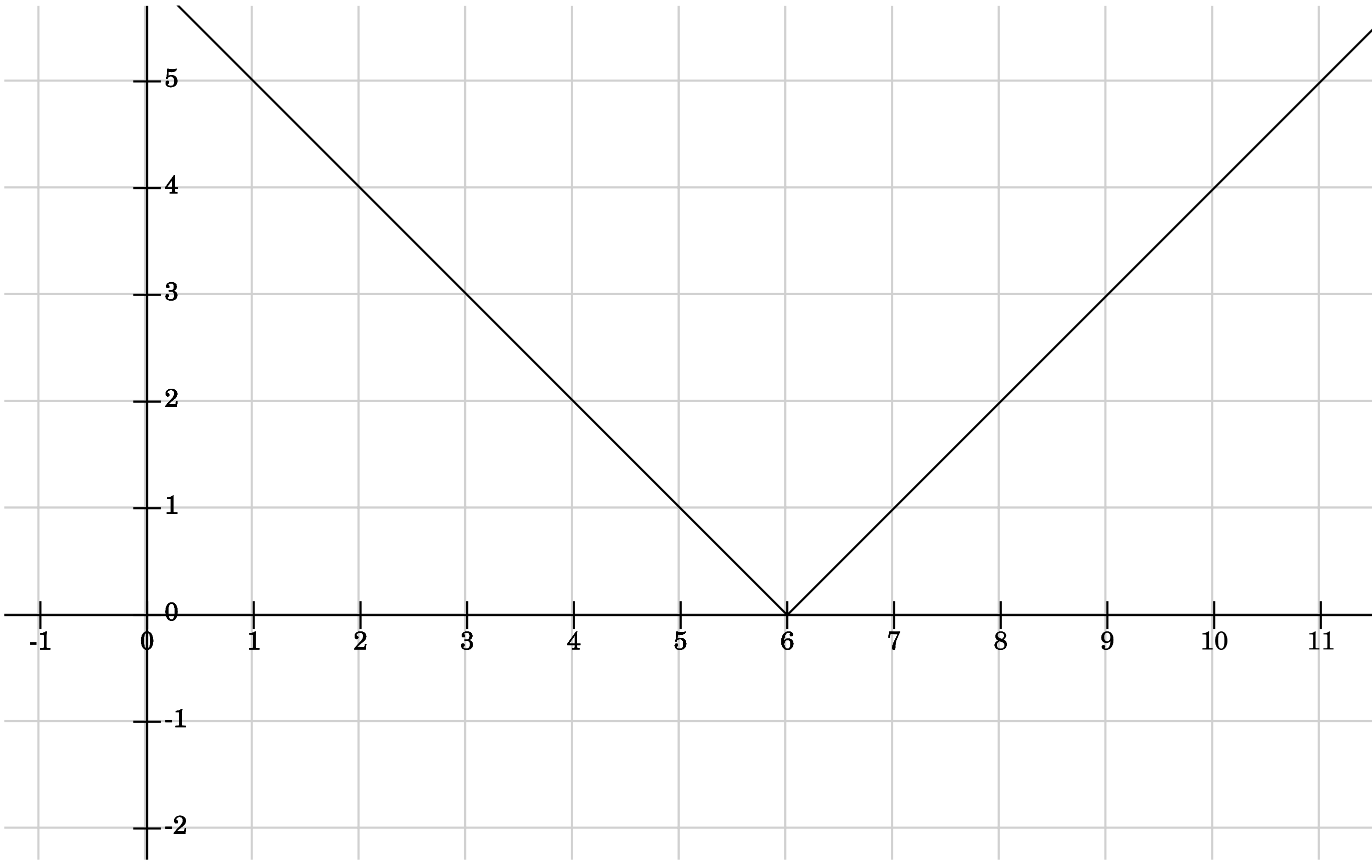


(b) Now consider $g(x) = |x - 6|$. Write out the piece-wise defined definition of this function carefully. THEN use that definition to graph the function g. Discuss how this graph relates to the graph of $f(x) = |x|$. Discuss how this function behaves near $x = 6$.

$$g(x) = |x - 6| = \begin{cases} x - 6 & \text{if } x - 6 \geq 0 \\ -(x - 6) & \text{if } x - 6 < 0 \end{cases} = \begin{cases} x - 6 & \text{if } x \geq 6 \\ 6 - x & \text{if } x < 6 \end{cases}$$

Domain=$\mathbb{R}$ Range=$\{y : y \geq 0\}$

This appears to be a shift of the graph $|x|$ to the right 6 units.

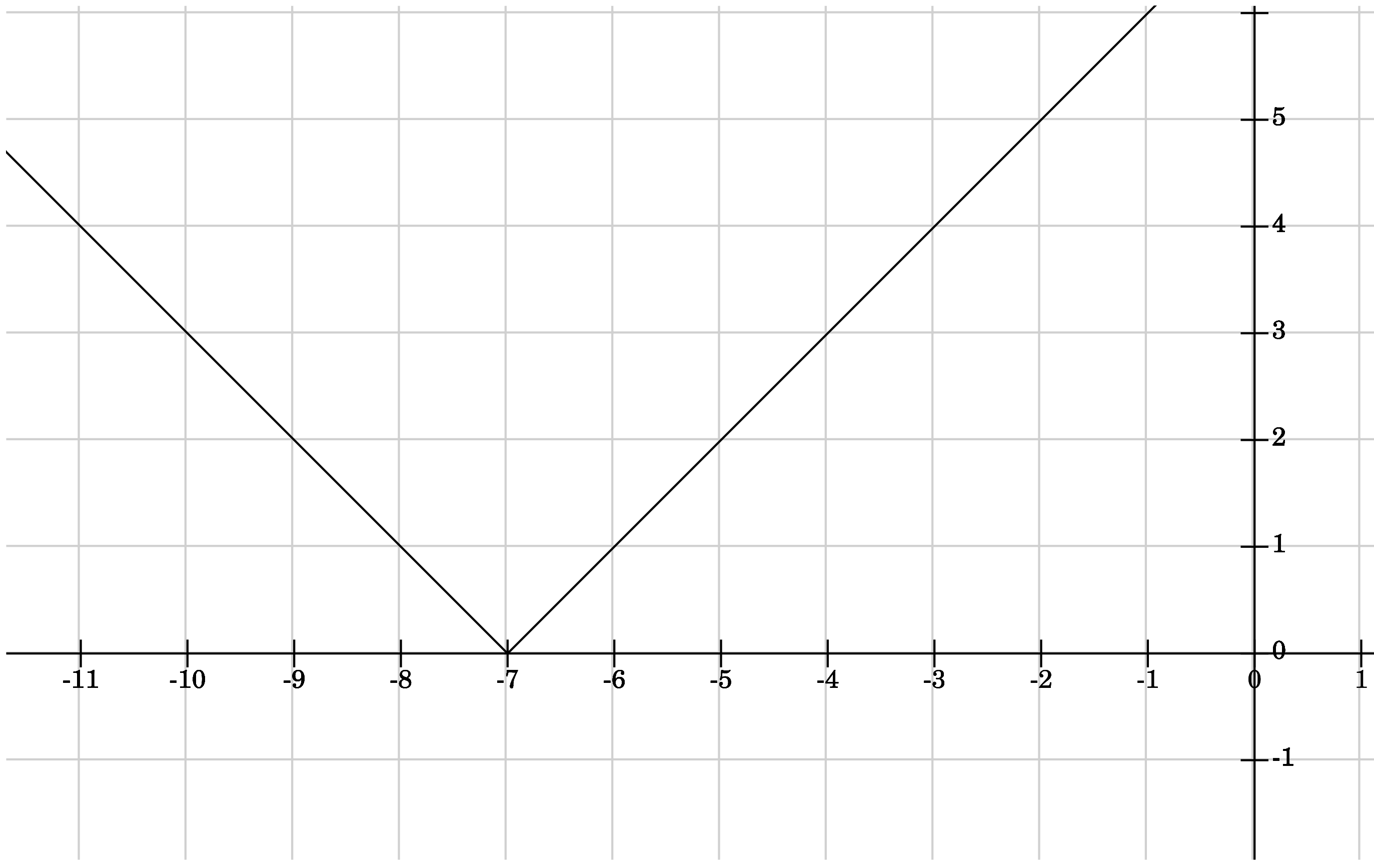


(c) Now consider $h(x) = |x+7|$. Write out the piece-wise defined definition of this function carefully. THEN use that definition to graph the function h. Discuss how this graph relates to the graph of $f(x) = |x|$. Discuss how this function behaves near $x = -7$.

$$h(x) = |x+7| = \begin{cases} x+7 & \text{if } x+7 \geq 0 \\ -(x+7) & \text{if } x+7 < 0 \end{cases} = \begin{cases} x+7 & \text{if } x \geq -7 \\ -x-7 & \text{if } x < -7 \end{cases}$$

Domain=$\mathbb{R}$ Range=$\{y : y \geq 0\}$

This appears to be a shift of the graph $|x|$ to the left 7 units.



6. Find the equation of the line L that passes through the point $(3, -1)$ and is **perpendicular** to the line $2x + 5y = 6$. THEN, does this new line L pass through the point $(1, -6)$?

   First simplify $2x + 5y = 6$ into slope-intercept form.

   $2x + 3y = 6 \implies 5y = -2x + 6 \implies y = -\frac{2}{5}x + \frac{6}{5}$. This line has slope $-\frac{2}{5}$. Then the perpendicular line has slope $\frac{5}{2}$.

   We now use the point $(3, -1)$ and the slope $\frac{5}{2}$ in point-slope form.

   $y - (-1) = \frac{5}{2}(x - 3)$ which simplifies to $y = \frac{5}{2}x - \frac{15}{2} - 1 = \frac{5}{2}x - \frac{15}{2} - \frac{2}{2} = \boxed{\frac{5}{2}x - \frac{17}{2}}$. Watch the algebra

   $\boxed{\text{Yes}}$, this point $(1, -6)$ lies on this line since $y(1) = \frac{5}{2}(1) - \frac{17}{2} = -\frac{12}{2} = -6$

7. Consider the function $f(x) = x^2 - 6x - 7$. Compute and **simplify** each of the following.

   (a) $f(0) = \boxed{-7}$

   (b) $f(-3) = (-3)^2 - 6(-3) - 7 = 9 + 18 - 7 = 27 - 7 = \boxed{20}$

   (c) $f(1) = 1^2 - 6(1) - 7 = 1 - 6 - 7 = \boxed{-12}$

   (d) For what values x does $f(x) = 0$? $f(x) = x^2 - 6x - 7 = (x - 7)(x + 1) = 0$ when $\boxed{x = 7 \text{ or } x = -1}$.

(e) $f(a) = \boxed{a^2 - 6a - 7}$

(f) $f(a+h) = (a+h)^2 - 6(a+h) - 7 = \boxed{a^2 + 2ah + h^2 - 6a - 6h - 7}$

(g) $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(a^2 + 2ah + h^2 - 6a - 6h - 7) - (a^2 - 6a - 7)}{h}$

$= \dfrac{a^2 + 2ah + h^2 - 6a - 6h - 7 - a^2 + 6a + 7}{h} = \dfrac{2ah + h^2 - 6h}{h} = \dfrac{h(2a + h - 6)}{h} = \boxed{2a + h - 6}$

(h) CHALLENGE!!! Compute $f(f(x))$. Show that it equals $x^4 - 12x^3 + 16x^2 + 120x + 84$. Yes... simplify! Come on you can try it...

$f(f(x)) = (x^2 - 6x - 7)^2 - 6(x^2 - 6x - 7) - 7 = (x^2 - 6x - 7)(x^2 - 6x - 7) - 6(x^2 - 6x - 7) - 7 = x^4 - 6x^3 - 7x^2 - 6x^3 + 36x^2 + 42x - 7x^2 + 42x + 49 - 6x^2 + 36x + 42 - 7 = \boxed{x^4 - 12x^3 + 16x^2 + 120x + 84}$

8. Consider the function defined by

$$f(x) = \begin{cases} x + 2 & \text{if} \quad x > 2 \\ -3 & \text{if} \quad x = 2 \\ x^2 & \text{if} \quad x = -1 < x < 2 \\ 5 & \text{if} \quad x < -1 \end{cases}$$

Graph $f(x)$ and find its Domain and Range. See me for a sketch.

$\boxed{\text{Domain}=\{x : x \neq -1\} \qquad \text{Range}=\{-3\} \cup \{y : y \geq 0 \text{ but } \neq 4\}}$

9. Consider the function defined piece-wise by

$$g(x) = \begin{cases} \dfrac{1}{x} & \text{if} \quad x > 0 \\ -\dfrac{1}{2}x + 1 & \text{if} \quad -4 < x \leq 0 \\ x^2 & \text{if} \quad x \leq -4 \end{cases}$$

Graph $g(x)$ and find its Domain and Range. See me for a sketch.

$\boxed{\text{Domain}=\mathbb{R} \qquad \text{Range}=\{y : y > 0\}}$