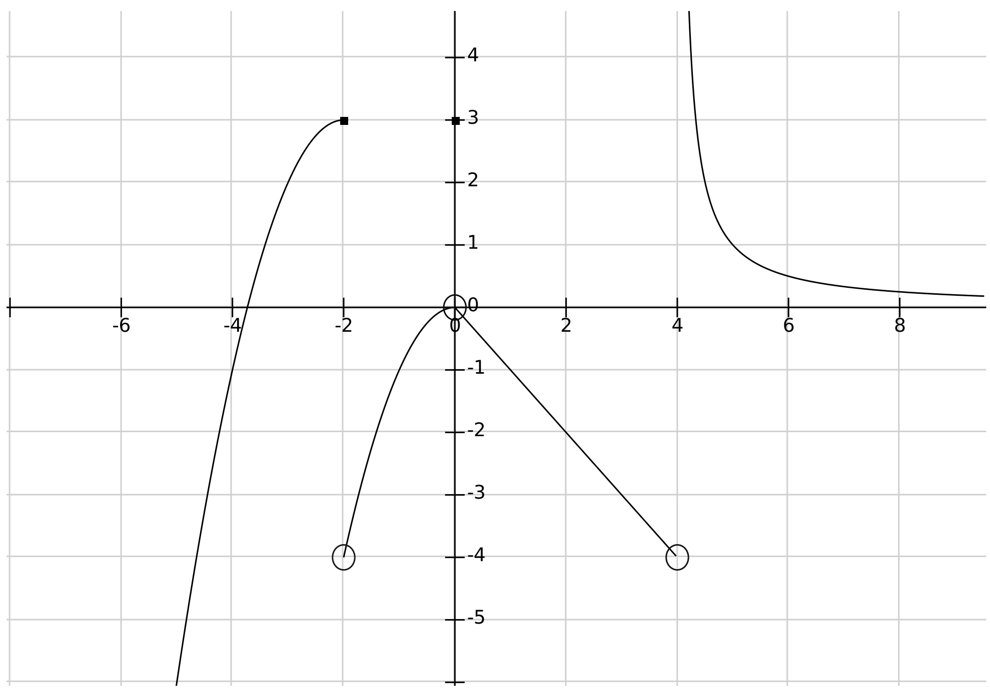
Summer Academy, Calculus with Algebra, 2019 Answer Key

Another Practice Exam #2, Friday, July 5, 2019

1. Explain why the given function is discontinuous at each of the given values a. Sketch the graph of the function.

$$f(x) = \begin{cases} \dfrac{1}{x-4} & \text{if } x > 4 \\ -x & \text{if } 0 < x < 4 \\ 3 & \text{if } x = 0 \\ -x^2 & \text{if } -2 < x < 0 \\ 3-(x+2)^2 & \text{if } x \leq -2 \end{cases} \qquad \text{with } a = -2,\ a = 0, \text{ and } a = 4.$$



Here, $f(x)$ is discontinuous (infinite discontinuity) at $a = 4$ because $\lim_{x\to 4} f(x)$ DNE because RHL≠LHL.

RHL: $\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \frac{1}{x-4} = +\infty$

LHL: $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} -x = -4$

Next, $f(x)$ is discontinuous (removable discontinuity) at $a = 0$ because although $\lim_{x\to 0} f(x) = 0$ and $f(0) = 3$ is defined, those two are not equal. That is, $0 = \lim_{x\to 0} f(x) \neq f(0) = 3$.

Finally, $f(x)$ is discontinuous (jump discontinuity) at $a = -2$ because $\lim_{x\to -2} f(x)$ DNE because RHL≠LHL.

RHL: $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} -x^2 = -4$

LHL: $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} 3 - (x+2)^2 = 3$

2. Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

   (a) $\lim_{x\to 7} \frac{x^2 - 6x - 7}{x^2 - 14x + 49} = \lim_{x\to 7} \frac{(x+1)(x-7)}{(x-7)^2} = \lim_{x\to 7} \frac{x+1}{x-7}$ DNE because RHL≠LHL

   RHL: $\lim_{x\to 7^+} \frac{x+1}{x-7} = \frac{8}{0^+} = +\infty$

   LHL: $\lim_{x\to 7^-} \frac{x+1}{x-7} = \frac{8}{0^-} = -\infty$

   (b) $$\lim_{x\to\infty} \frac{4x^2 + 8x - 1}{5x^2 - 7} \cdot \left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right) = \lim_{x\to\infty} \frac{4 + \frac{8}{x} - \frac{1}{x^2}}{5 - \frac{7}{x^2}} = \boxed{\frac{4}{5}}$$

   (c) $$\lim_{x\to\infty} \frac{3x^6 - 5}{9x^4 + 3x} \cdot \left(\frac{\frac{1}{x^4}}{\frac{1}{x^4}}\right) = \lim_{x\to\infty} \frac{3x^2 - \frac{5}{x^4}}{9 + \frac{3}{x^3}} = \boxed{+\infty}$$

   (d) $$\lim_{x\to\infty} \frac{2x^4 - 9x + 8}{x^9 + 3} \cdot \left(\frac{\frac{1}{x^9}}{\frac{1}{x^9}}\right) = \lim_{x\to\infty} \frac{\frac{2}{x^5} - \frac{9}{x^8} + \frac{8}{x^9}}{1 + \frac{3}{x^9}} = \frac{0}{1} = \boxed{0}$$

3. Compute the derivative of each of the following functions. For these problems, you do **not** need to simplify your derivative. You may use the quicker Differentiation Rules at this point, unless otherwise stated.

(a) $f(x) = \frac{5}{6}x + x^{\frac{5}{6}} + x^{\frac{6}{5}} + \frac{1}{x^{\frac{5}{6}}} + \frac{6}{5} + \frac{5}{6x^6} - \frac{6}{x^5}$

$= \frac{5}{6}x + x^{\frac{5}{6}} + x^{\frac{6}{5}} + x^{-\frac{5}{6}} + \frac{6}{5} + \frac{5}{6}x^{-6} - 6x^{-5}$

$$f'(x) = \boxed{\frac{5}{6} + \frac{5}{6}x^{-\frac{1}{6}} + \frac{6}{5}x^{\frac{1}{5}} - \frac{5}{6}x^{-\frac{11}{6}} + 0 - 5x^{-7} + 30x^{-6}}$$

(b) $y = \sqrt{\frac{x}{7} - \frac{1}{x^7}} = \sqrt{\frac{1}{7}x - x^{-7}}$

$$y' = \boxed{\frac{1}{2\sqrt{\frac{x}{7} - \frac{1}{x^7}}} \cdot \left(\frac{1}{7} + 7x^{-8}\right)}$$

(c) $f(x) = \left(x^4 - \frac{4}{x^4}\right)\left(4x + \frac{\sqrt{x}}{4}\right) = (x^4 - 4x^{-4})\left(4x + \frac{1}{4}\sqrt{x}\right)$

$$f'(x) = \boxed{\left(x^4 - \frac{4}{x^4}\right)\left(4 + \left(\frac{1}{4}\right)\frac{1}{2\sqrt{x}}\right) + \left(4x + \frac{\sqrt{x}}{4}\right)\left(4x^3 + \frac{16}{x^5}\right)}$$

it's not necessary, but we can simplify the function to match it the other method

$f'(x) = 4x^4 + \frac{x^{\frac{7}{2}}}{8} - \frac{16}{x^4} - \frac{1}{2x^{\frac{9}{2}}} + 16x^4 + \frac{64}{x^4} + x^{\frac{7}{2}} + \frac{4}{x^{\frac{9}{2}}} = 20x^4 + \frac{9}{8}x^{\frac{7}{2}} + \frac{48}{x^4} + \frac{7}{2x^{\frac{9}{2}}}$

OR you can simplify the function first and then use the power rules...

$f(x) = \left(x^4 - \frac{4}{x^4}\right)\left(4x + \frac{\sqrt{x}}{4}\right)$

$= 4x^5 + \frac{1}{4}x^{\frac{9}{2}} - \frac{16}{x^3} - \frac{1}{x^{\frac{7}{2}}}$

$= 4x^5 + \frac{1}{4}x^{\frac{9}{2}} - 16x^{-3} - x^{-\frac{7}{2}}$

$f'(x) = 20x^4 + \frac{9}{8}x^{\frac{7}{2}} + 48x^{-4} + \frac{7}{2}x^{-\frac{9}{2}}$

$= \boxed{20x^4 + \frac{9}{8}x^{\frac{7}{2}} + \frac{48}{x^4} + \frac{7}{2}\frac{1}{x^{\frac{9}{2}}}}$

(d) $y = \left(\frac{1}{3x^8} + 8x^3\right)^{\frac{3}{8}} = \left(\frac{1}{3}x^{-8} + 8x^3\right)^{\frac{3}{8}}$

$$y' = \boxed{\frac{3}{8}\left(\frac{1}{3x^8} + 8x^3\right)^{-\frac{5}{8}}\left(-\frac{8}{3x^9} + 24x^2\right)}$$

4. Compute the derivative of $f(x) = \dfrac{6-5x}{2+4x}$ **two** different ways:

- First use the **limit definition of the derivative**.
- Second use the Quotient Rule.

Next, simplify your answer in the first part. Then compute the second derivative $f''(x)$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{6-5(x+h)}{2+4(x+h)} - \dfrac{6-5x}{2+4x}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{[6-5x-5h](2+4x)-(6-5x)[2+4x+4h]}{(2+4(x+h))(2+4x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{12-10x-10h+24x-20x^2-20xh-(12+24x+24h-10x-20x^2-20xh)}{(2+4(x+h))(2+4x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{12-10x-10h+24x-20x^2-20xh-12-24x-24h+10x+20x^2+20xh}{(2+4(x+h))(2+4x)}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\dfrac{-34h}{(2+4(x+h))(2+4x)}\right)}{h} = \lim_{h\to 0} \frac{-34h}{(2+4(x+h))(2+4x)}\left(\frac{1}{h}\right)$$

$$= \lim_{h\to 0} \frac{-34}{(2+4(x+h))(2+4x)} = \boxed{\frac{-34}{(2+4x)^2}}$$

Other method, Quotient Rule:

$$f'(x) = \frac{(2+4x)(-5)-(6-5x)(4)}{(2+4x)^2} = \frac{-10-20x-24+20x}{(2+4x)^2} = \boxed{\frac{-34}{(2+4x)^2}} \text{ match!}$$

Next, simplify your answer in the first part. Then compute the second derivative $f''(x)$.

$f'(x) = (-34)(2+4x)^{-2}$

$f''(x) = (-34)(-2)(2+4x)^{-3}(4) = \boxed{\dfrac{272}{(2+4x)^3}}$

5. Find **all** x-coordinates at which the graphs of the following functions have horizontal tangent lines. Please **simplify** your derivatives first. Why?

(a) $f(x) = \dfrac{(4x+3)^3}{(8x+2)^4}$

$$f'(x) = \frac{(8x+2)^4 \cdot 3(4x+3)^2(4) - (4x+3)^3 \cdot 4(8x+2)^3(8)}{\left((8x+2)^4\right)^2}$$

$$= \frac{4(8x+2)^3(4x+3)^2\left[3(8x+2)-8(4x+3)\right]}{(8x+2)^8}$$
$$= \frac{4(8x+2)^3(4x+3)^2\left[24x+6-32x-24\right]}{(8x+2)^8}$$
$$= \frac{4(4x+3)^2\left[-8x-18\right]}{(8x+2)^5} \overset{\text{set}}{=} 0$$

For a product of terms in the numerator to equal 0, then any of the terms could be equal to 0. Therefore,

$4x+3=0 \quad$ OR $\quad -8x-18=0$

Finally, solving each of the equations

$\boxed{x=-\frac{3}{4}}$ OR $\boxed{x=-\frac{9}{4}}$

(b) $f(x)=(x+1)^2\cdot\sqrt{x+2}$

$$f'(x)=(x+1)^2\frac{1}{2\sqrt{x+2}}(1)+\sqrt{x+2}(2)(x+1)(1)$$
$$=(x+1)^2\frac{1}{2\sqrt{x+2}}+\sqrt{x+2}(2)(x+1)\cdot\left(\frac{2\sqrt{x+2}}{2\sqrt{x+2}}\right) \text{ common denominator}$$
$$=\frac{(x+1)^2}{2\sqrt{x+2}}+\frac{(x+2)(4)(x+1)}{2\sqrt{x+2}}$$
$$=\frac{(x+1)^2+4(x+2)(x+1)}{2\sqrt{x+2}}=\frac{x^2+2x+1+4x^2+12x+8}{2\sqrt{x+2}}$$
$$=\frac{5x^2+14x+9}{2\sqrt{x+2}} \overset{\text{set}}{=} 0$$

Therefore, clearing the denominator $5x^2+14x+9=0$.

Factor, $5x^2+14x+9=(5x+9)(x+1)=0$. So $\boxed{x=-\frac{9}{5}}$ OR $\boxed{x=-1}$

6. Compute the derivative of $f(x)=\dfrac{x}{x-1}+\dfrac{x}{x+1}$. Simplify your answer to a single fraction.

We have two options. You can first simplify the function to a single fraction and then differentiate. Or you can differentiate first and then simplify.

First let's simplify $f(x)=\dfrac{x}{x-1}+\dfrac{x}{x+1}=\left(\dfrac{x+1}{x+1}\right)\dfrac{x}{x-1}+\dfrac{x}{x+1}\left(\dfrac{x-1}{x-1}\right)=\dfrac{x(x+1)+x(x-1)}{(x-1)(x+1)}=\dfrac{x^2+x+x^2-x}{(x-1)(x+1)}=\dfrac{2x^2}{x^2-1}$.

Now let's differentiate

$$f'(x) = \frac{(x^2-1)(4x) - 2x^2(2x)}{(x^2-1)^2} = \frac{4x^3 - 4x - 4x^3}{(x^2-1)^2} = \boxed{\frac{-4x}{(x^2-1)^2}}$$

OR let's differentiate first...and then find a common denominator

$$f'(x) = \frac{(x-1)(1) - x(1)}{(x-1)^2} + \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{-1}{(x-1)^2} + \frac{1}{(x+1)^2}$$

$$= \frac{-(x+1)^2 + (x-1)^2}{(x-1)^2(x+1)^2} = \frac{-(x^2+2x+1) + x^2 - 2x + 1}{(x-1)^2(x+1)^2}$$

$$= \frac{-x^2 - 2x - 1 + x^2 - 2x + 1}{(x-1)^2(x+1)^2} = \frac{-4x}{(x-1)^2(x+1)^2} = \boxed{\frac{-4x}{(x^2-1)^2}}$$

7. Find the equation of the tangent line to the curve $x^3 + x^2y = 6 - 4y^2$ at the point $(1,1)$.

   First, implicitly differentiate by taking the derivative of both sides:

   $$\frac{d}{dx}\left(x^3 + x^2y\right) = \frac{d}{dx}\left(6 - 4y^2\right)$$

   $$3x^2 + x^2\frac{dy}{dx} + y(2x) = 0 - 8y\frac{dy}{dx}$$

   We can plug the point $(1,1)$ immediately and solve for the derivative value $\frac{dy}{dx}$

   $$3(1)^2 + (1)^2\frac{dy}{dx} + (1)(2(1)) = 0 - 8(1)\frac{dy}{dx}$$

   $$3 + \frac{dy}{dx} + 2 = -8\frac{dy}{dx}$$

   Isolate and solve for $\frac{dy}{dx}$

   $$9\frac{dy}{dx} = -5$$

   OR you can solve for $\frac{dy}{dx}$ and then plug in $x = 1$ and $y = 1$.

   Finally, $\frac{dy}{dx} = -\frac{5}{9}$.

   Now the equation of the tangent line at the point $(1,1)$ with slope $-\frac{5}{9}$ is given by

   $$y - 1 = -\frac{5}{9}(x-1)$$

   or $y - 1 = -\frac{5}{9}x - \frac{5}{9}$

or $y = -\frac{5}{9}x - \frac{5}{9} + 1$

or $y = -\frac{5}{9}x + \frac{5}{9} + \frac{9}{9}$

or $y = \boxed{-\frac{5}{9}x + \frac{14}{9}}$