Practice Exam #2 Summer Academy 2019 Answer Key

1.a $\lim_{x\to 7} g(x) \underset{\substack{\uparrow\\ g \text{ continuous}\\ @x=7}}{=} g(7) \underset{\substack{\uparrow\\ \text{given}}}{=} \boxed{-3}$

b. $g\circ f(5) = g(\cancel{f(5)}^{\,7}) \underset{\substack{\uparrow\\ \text{given}}}{=} g(7) \underset{\substack{\uparrow\\ \text{given}}}{=} \boxed{-3}$

c. No $f(7) \neq 5$ because by assumption $f(x)$ was assumed to <u>NOT</u> be continuous at $x=7$, so by definition of continuity

$f(7)$ is undefined (no value given) or $\lim_{x\to 7} f(x) \neq f(7)$

Here $\lim_{x\to 7} f(x) = 5$ was given so $f(7) \neq 5 = \lim_{x\to 7} f(x)$.

2.a. $\lim_{x\to 3} \frac{x^2-5x+6}{x^2-6x+9} \overset{0/0}{=} \lim_{x\to 3} \frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}(x-3)} = \lim_{x\to 3} \frac{x-2}{x-3}$ $\overset{1/0}{}$ Sign Analysis. ⟨Analyze fully What kind of blow up?⟩

RHL: $\lim_{x\to 3^+} \frac{x-2}{x-3} = \frac{1}{3^+-3} = \frac{1}{0^+} = +\infty$

DNE b/c RHL $\neq$ LHL.

LHL: $\lim_{x\to 3^-} \frac{x-2}{x-3} = \frac{1}{3^--3} = \frac{1}{0^-} = -\infty$

$$2.b.\ \lim_{x\to\infty} \frac{3x^2-5x+6}{7-5x^2}\left(\frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}\right) = \lim_{x\to\infty} \frac{3-\cancel{5/x}^{\,0}+\cancel{\frac{6}{x^2}}^{\,0}}{\cancel{\frac{7}{x^2}}_{\,0}-5} = \boxed{\frac{-3}{5}}$$

$$2c.\ \lim_{x\to\infty} \frac{4x^2-5}{3x^7+9}\left(\frac{\left(\frac{1}{x^7}\right)}{\left(\frac{1}{x^7}\right)}\right) = \lim_{x\to\infty} \frac{\cancel{4/x^5}^{\,0}-\cancel{5/x^7}^{\,0}}{3+\cancel{9/x^7}^{\,0}} = \frac{0}{3} = \boxed{0}$$

$$2d.\ \lim_{x\to\infty} \frac{6x^3-4}{x+5}\left(\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}\right) = \lim_{x\to\infty} \frac{\cancel{6x^2}^{\,\infty}-\cancel{4/x}^{\,0}}{\underbrace{1}_{\text{"rescue"}}+\cancel{5/x}^{\,0}} = \boxed{\infty}$$

$$3a.\ y=\frac{5}{6}x+x^{5/6}+x^{-5/6}+(5x+6)^{1/2}+(5x+6)^{-1/2}$$

$$\boxed{y'=5/6+\frac{5}{6}x^{-1/6}-5/6\,x^{-11/6}+\frac{1}{2}(5x+6)^{-1/2}(5)-\frac{1}{2}(5x+6)^{-3/2}(5)}$$

$$b.\ y=\left[\frac{2\sqrt{x}+x^3}{x^{2/3}+\frac{2}{3}x}\right]^{2/3}$$

$$\boxed{y'=\frac{2}{3}\left[\frac{2\sqrt{x}+x^3}{x^{2/3}+\frac{2}{3}x}\right]^{-1/3}\cdot\frac{\left(x^{2/3}+\frac{2}{3}x\right)\left[\cancel{2}\cdot\frac{1}{\cancel{2}\sqrt{x}}+3x^2\right]-\left(2\sqrt{x}+x^3\right)\left[\frac{2}{3}x^{-1/3}+\frac{2}{3}\right]}{\left(x^{2/3}+\frac{2}{3}x\right)^2}}$$

3.(c) $f(x) = \left(\overset{3x^{-2}}{\frac{3}{x^2}} - \overset{2x^{-3}}{\frac{2}{x^3}}\right)^9 \sqrt{5-x^2}$

$$\boxed{f'(x) = \left(\frac{3}{x^2}-\frac{2}{x^3}\right)^9\left(\frac{1}{2\sqrt{5-x^2}}\right)(-2x) + \sqrt{5-x^2}\,(9)\left(\frac{3}{x^3}-\frac{2}{x^3}\right)^8\left[-6x^{-3}+6x^{-4}\right]}$$

(d) $y = \dfrac{\frac{1}{x}-6x^3}{\sqrt{7x+x^8}}$

$$\boxed{y' = \frac{\sqrt{7x+x^8}\left(-\frac{1}{x^2}-18x^2\right) - \left(\frac{1}{x}-6x^3\right)\frac{1}{2\sqrt{7x+x^8}}\left(7+8x^7\right)}{7x+x^8}}$$

4. $y = \left(6x+\sqrt{8+x^2}\right)^{3/2}$ $\quad y(1) = \left(6+\sqrt{9}\right)^{3/2} = \left(\sqrt{9}\right)^3 = 3^3 = 27$

$y' = \frac{3}{2}\left(6x+\sqrt{8+x^2}\right)^{1/2}\cdot\left[6+\frac{1}{\cancel{2}\sqrt{8+x^2}}(\cancel{2}x)\right]$

$y'(1) = \frac{3}{2}\sqrt{6+\sqrt{9}\to 3}\cdot\left[6+\frac{1}{\sqrt{9}\to 3}\right]$

$= \frac{3}{2}\sqrt{9}\to 3\cdot\left[\frac{18}{3}+\frac{1}{3}\right]$

$= \frac{9}{2}\cdot\frac{19}{3} = \frac{57}{2}$ ← specific slope

Point $(1, 27)$ Slope $y'(1) = 57/2$

Equation of Tangent Line

$y - 27 = 57/2\,(x-1)$ $\quad y - 27 = \frac{57}{2}x - \frac{57}{2}$ $\quad \boxed{y = \frac{57}{2}x - \frac{3}{2}}$

$+27$ $\quad +27 \to 54/2$

5. $f(x) = \dfrac{3x-1}{2-5x}$

(1) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\dfrac{3(x+h)-1}{2-5(x+h)} - \dfrac{3x-1}{2-5x}}{h}$

$= \lim\limits_{h \to 0} \dfrac{(2-5x)(3x+3h-1) - (3x-1)(2-5x-5h)}{(2-5x-5h)(2-5x)} \cdot \left(\dfrac{1}{h}\right)$

$= \lim\limits_{h \to 0} \dfrac{6x+6h-2-15x^2-15xh+5x - (6x-15x^2-15xh-2+5x+5h)}{(2-5x-5h)(2-5x)} \cdot \left(\dfrac{1}{h}\right)$

$= \lim\limits_{h \to 0} \dfrac{\cancel{6x}+\boxed{6h}\cancel{-2}\cancel{-15x^2}\cancel{-15xh}\cancel{+5x}\cancel{-6x}\cancel{+15x^2}\cancel{+15xh}\cancel{+2}\cancel{-5x}\boxed{-5h}}{(2-5x-5h)(2-5x)} \cdot \left(\dfrac{1}{h}\right)$

$= \lim\limits_{h \to 0} \dfrac{\cancel{h}}{(2-5x-\underbrace{5h}_{0})(2-5x)} \cdot \dfrac{1}{\cancel{h}} = \boxed{\dfrac{1}{(2-5x)^2}}$

(2) $f'(x) = \dfrac{(2-5x)(3) - (3x-1)(-5)}{(2-5x)^2} = \dfrac{6\cancel{-15x}\cancel{+15x}-5}{(2-5x)^2} = \boxed{\dfrac{1}{(2-5x)^2}}$

Simplify $f'(x) = (2-5x)^{-2}$

$\Longrightarrow \quad f''(x) = -2(2-5x)^{-3}(-5) = \boxed{\dfrac{10}{(2-5x)^3}}$

6. $f(x) = (5+3x^2)^8 (7-x^2)^3$

$$f'(x) = (5+3x^2)^8 \cdot 3(7-x^2)^2(-2x) + (7-x^2)^3 \cdot 8(5+3x^2)^7(6x)$$

$$= x(5+3x^2)^7(7-x^2)^2 6\left[(-1)(5+3x^2) + 8(7-x^2)\right]$$

$$-5 - 3x^2 + 56 - 8x^2$$

$$= x(5+3x^2)^7(7-x^2)^2 \cdot 6\left[-11x^2+51\right] \overset{set}{=} 0$$

$\Rightarrow \boxed{x=0}$ or $5+3x^2=0$ or $7-x^2=0$ or $-11x^2+51=0$

$3x^2=-5$ No Solution

$x^2=7$, $\boxed{x=\pm\sqrt{7}}$

$x^2 = 51/11$, $\boxed{x = \pm\sqrt{51/11}}$

7. $f(x) = \dfrac{5x}{1+x}$

$$f'(x) = \frac{(1+x)(5) - 5x(1)}{(1+x)^2} = \frac{5}{(1+x)^2}$$

$$f'(0) = \frac{5}{(1+0)^2} = \boxed{5}$$

$$f'(1) = \frac{5}{(1+1)^2} = \boxed{\frac{5}{4}}$$

$$f'(2) = \frac{5}{(1+2)^2} = \boxed{\frac{5}{9}}$$

8. (a) $\dfrac{x}{y+1} = x^2 - y^2$

$$\frac{d}{dx}\left[\frac{x}{y+1}\right] = \frac{d}{dx}\left[x^2 - y^2\right]$$ Implicitly differentiate both sides

$$\frac{(y+1)(1) - x \cdot \frac{dy}{dx}}{(y+1)^2} = 2x - 2y\frac{dy}{d}$$ plug in $x=1, y=0$

$$\frac{1 - \frac{dy}{dx}}{\cancel{1^2}\ 1} = 2 - 0$$

$$1 - \frac{dy}{dx} = 2$$

$$-\frac{dy}{dx} = 1 \Rightarrow \left.\frac{dy}{dx}\right|_{(1,0)} = -1$$ specific slope

Equation of Tangent Line.

$$y - 0 = -1(x-1)$$

$$\boxed{y = -x + 1}$$

8(b). $4(x+y)^2 = x^2y^2$

$$\frac{d}{dx}\left[4(x+y)^2\right] = \frac{d}{dx}\left[x^2y^2\right]$$ implicitly differentiate

$$8(x+y)\left[1+\frac{dy}{dx}\right] = x^2 \cdot 2y\,\frac{dy}{dx} + y^2(2x)$$ plug in $(-2,1)$

$$8(-2+1)\left[1+\frac{dy}{dx}\right] = (4)\cdot 2\,\frac{dy}{dx} - 4$$

$$-8\left[1+\frac{dy}{dx}\right] = 8\,\frac{dy}{dx} - 4$$

$$-8 - 8\frac{dy}{dx} = 8\frac{dy}{dx} - 4$$

$$-4 = 16\frac{dy}{dx}$$

$$\left.\frac{dy}{dx}\right|_{(-2,1)} = \frac{-1}{4}$$ ← specific slope

Equation of Tangent Line

$$y-1 = -\tfrac{1}{4}(x-(-2))$$

$$y-1 = -\tfrac{1}{4}(x+2)$$

$$y-1 = -\tfrac{1}{4}x - \tfrac{1}{2}$$

$$+1 \qquad\qquad +1$$

$$\boxed{y = -\tfrac{1}{4}x + \tfrac{1}{2}}$$