Name: Answer Key

Summer Academy

Midterm Exam #2

July 6, 2019

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, or webpages, or other aids are permitted.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

Problem	Score	Possible Points
1		20
2		28
3		8
4		18
5		10
6		16
Total		100

1. [20 Points] Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim_{x\to 3} \frac{x^2-5x+6}{x^2-6x+9} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}(x-3)} = \lim_{x\to 3} \frac{x-2}{x-3}$ DNE b/c RHL ≠ LHL

RHL: $\lim_{x\to 3^+} \frac{x-2}{x-3} = \frac{1}{0^+} = +\infty$

LHL: $\lim_{x\to 3^-} \frac{x-2}{x-3} = \frac{1}{0^-} = -\infty$

(b) $\lim_{x\to\infty} \frac{3x^2-5x+6}{7-5x^2} \cdot \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{3-\cancel{5/x}^{\to 0}+\cancel{6/x^2}^{\to 0}}{\underbrace{\cancel{7/x^2}}_{0}-5} = \boxed{-\frac{3}{5}}$

(c) $\lim_{x\to\infty} \frac{4x^2-5}{3x^7+9} \cdot \frac{\left(\frac{1}{x^7}\right)}{\left(\frac{1}{x^7}\right)} = \lim_{x\to\infty} \frac{\cancel{\frac{4}{x^5}}^{\to 0}-\cancel{\frac{5}{x^7}}^{\to 0}}{3+\cancel{9/x^7}_{\to 0}} = \frac{0}{3} = \boxed{0}$

(d) $\lim_{x\to\infty} \frac{6x^3-4}{x+5} \cdot \frac{\left(1/x\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{\cancel{6x^2}^{\infty}-\cancel{\frac{4}{x}}^{\to 0}}{1+\cancel{5/x}_{\to 0}} = \boxed{\infty}$

2. [28 Points] Compute the derivative of each of the following functions. For these problems, you do **NOT** need to simplify your derivative.

(a) $y = \frac{5}{6}x + x^{\frac{5}{6}} + \frac{1}{x^{\frac{5}{6}}} + \sqrt{5x+6} + \frac{1}{\sqrt{5x+6}}$

$= \frac{5}{6}x + x^{5/6} + x^{-5/6} + \sqrt{5x+6} + (5x+6)^{-1/2}$

$$\boxed{y' = \frac{5}{6} + \frac{5}{6}x^{-1/6} - \frac{5}{6}x^{\frac{-11}{6}} + \frac{1}{2\sqrt{5x+6}} \cdot (5) - \frac{1}{2}(5x+6)^{-3/2}(5)}$$

(b) $y = \left(\frac{2\sqrt{x} + x^3}{x^{\frac{2}{3}} + \frac{2}{3}x} \right)^{\frac{2}{3}}$

$$\boxed{y' = \frac{2}{3}\left(\frac{2\sqrt{x}+x^3}{x^{2/3}+\frac{2}{3}x}\right)^{-1/3} \cdot \left[\frac{\left(x^{2/3}+\frac{2}{3}x\right)\left(2\cdot\frac{1}{2\sqrt{x}} + 3x^2\right) - \left(2\sqrt{x}+x^3\right)\left(\frac{2}{3}x^{-1/3}+\frac{2}{3}\right)}{\left(x^{2/3}+\frac{2}{3}x\right)^2}\right]}$$

$3x^{-2} - 2x^{-3}$

2. (Continued) Compute the derivative of each of the following functions. For these problems, you do **NOT** need to simplify your derivative.

(c) $f(x) = \left(\dfrac{3}{x^2} - \dfrac{2}{x^3}\right)^9 \sqrt{5 - x^2}$

$$\boxed{f'(x) = \left(\frac{3}{x^2} - \frac{2}{x^3}\right)^9 \cdot \frac{1}{2\sqrt{5-x^2}}(-2x) + \sqrt{5-x^2} \cdot 9\left(\frac{3}{x^2} - \frac{2}{x^3}\right)^8 \left(-6x^{-3} + 6x^{-4}\right)}$$

$6x^{-1}$

(d) $y = \dfrac{\frac{1}{x} - 6x^3}{\sqrt{7x + x^8}}$

$$y' = \frac{\sqrt{7x+x^8}\left(-x^{-2} - 18x^2\right) - \left(\frac{1}{x} - 6x^3\right) \cdot \frac{1}{2\sqrt{7x+x^8}} \cdot (7 + 8x^7)}{\left(\sqrt{7x+x^8}\right)^2}$$

$7x + x^8$

3. [8 Points] Find the equation of the tangent line to this curve $y = \left(6x + \sqrt{8+x^2}\right)^{\frac{3}{2}}$ at the point where $x = 1$.

Note: $y(1) = (6 + \underbrace{\sqrt{9}}_{3})^{3/2} = 9^{3/2} = (\sqrt{9})^3 = 3^3 = 27$

Point: $(1, 27)$

$$y' = \frac{3}{2}\left(6x + \sqrt{8+x^2}\right)^{1/2}\left[6 + \frac{1}{2\sqrt{8+x^2}}(2x)\right]$$

$$y'(1) = \frac{3}{2}(6 + \underbrace{\sqrt{9}}_{3})^{1/2}\left[6 + \frac{1}{\cancel{2}\sqrt{9}} \cdot \cancel{2}\right]$$

$\sqrt{9} = 3$

$$= \frac{9}{2}\left[6 + \frac{1}{3}\right] = \frac{9}{2}\left[\frac{18}{3} + \frac{1}{3}\right] = \frac{\overset{3}{\cancel{9}}}{2}\left[\frac{19}{\cancel{3}}\right] = \boxed{\frac{57}{2}}$$

Point Slope Form

$$y - 27 = \frac{57}{2}(x - 1)$$

$\frac{54}{2}$

$$y = \frac{57}{2}x - \frac{57}{2} + 27$$

$$\boxed{y = \frac{57}{2}x - \frac{3}{2}}$$ ← Equation of Tangent Line

4. [18 Points]

(a) Compute the derivative of $f(x) = \dfrac{3x-1}{2-5x}$ **two different ways:**

- **First use the limit definition of the derivative.**
- **Second use the Quotient Rule.**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{\dfrac{3(x+h)-1}{2-5(x+h)} - \dfrac{3x-1}{2-5x}}{h}$$

$$= \lim_{h \to 0} \frac{\left(\dfrac{(3x+3h-1)(2-5x) - (3x-1)(2-5x-5h)}{(2-5(x+h))(2-5x)}\right)}{\dfrac{h}{1}}$$

$$= \lim_{h \to 0} \frac{\cancel{6x} + 6h \cancel{-2} \cancel{-15x^2} - 15xh + \cancel{5x} \cancel{-6x} + \cancel{15x^2} + 15xh \cancel{+2} \cancel{-5x} - 5h}{h \cdot (2-5(x+h))(2-5x)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(2-5(x+h))(2-5x)} = \lim_{h \to 0} \frac{1}{(2-5(x+h))(2-5x)} = \boxed{\frac{1}{(2-5x)^2}}$$

Match!

OR

$$f'(x) = \frac{(2-5x)(3) - (3x-1)(-5)}{(2-5x)^2} = \frac{6 - \cancel{15x} + \cancel{15x} - 5}{(2-5x)^2} = \boxed{\frac{1}{(2-5x)^2}}$$

4. (Continued)

(b) Compute the derivative of $f(x) = \sqrt{3 - x + x^2}$ **two different ways:**

- First use the **limit definition of the derivative.**
- Second use the Chain Rule.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3-(x+h)+(x+h)^2} - \sqrt{3-x+x^2}}{h} \cdot \frac{\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2}}{\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2}}$$

don't drop

$$= \lim_{h \to 0} \frac{3-(x+h)+(x+h)^2 - (3-x+x^2)}{h\left(\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{3} \cancel{-x} - h + \cancel{x^2} + 2xh + h^2 \cancel{-3} \cancel{+x} \cancel{-x^2}}{h\left(\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2}\right)}$$

$$= \lim_{h \to 0} \frac{-h + 2xh + h^2}{h\left(\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2}\right)}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-1 + 2x \cancel{+h})}{\cancel{h}\left(\sqrt{3-(x+h)+(x+h)^2} + \sqrt{3-x+x^2}\right)}$$

2 copies

$$= \boxed{\frac{-1+2x}{2\sqrt{3-x+x^2}}}$$

OR Chain Rule.

$$f'(x) = \boxed{\frac{1}{2\sqrt{3-x+x^2}} \cdot (-1+2x)}$$

Match!

5. [10 Points] Suppose that G and H are functions, and

- $\lim_{x\to 5} G(x) = 6$
- $\lim_{x\to -9} H(x) = -4$
- $\lim_{x\to 8} G(x) = 7$
- $G(x)$ is continuous at $x = 8$.
- $H(x)$ is continuous at $x = 7$.
- $G(5) = -9$
- $H(7) = -9$

Answer the following questions or evaluate the following quantities and fully **justify** your answers.

(a) Compute $G(8) = \lim_{x\to 8} G(x) = 7$

b/c G is continuous at $x=8$

given

(b) Compute $\lim_{x\to 7} H(x) = H(7) = -9$

b/c H is continuous at $x=7$

given

(c) Compute $H\circ G(8) = H(G(8)) = H(7) = -9$

$G(8) = 7$

See (a) above

given

(d) Does $H(-9) = -4$? Yes, No, or Not Enough Information? Why or why not?

We don't know, not enough information

We know $\lim_{x\to -9} H(x) = -4$, but we were not given whether $H(x)$ was continuous at $x=-9$ which would mean $\lim_{x\to -9} H(x) = H(-9)$ given as -4.

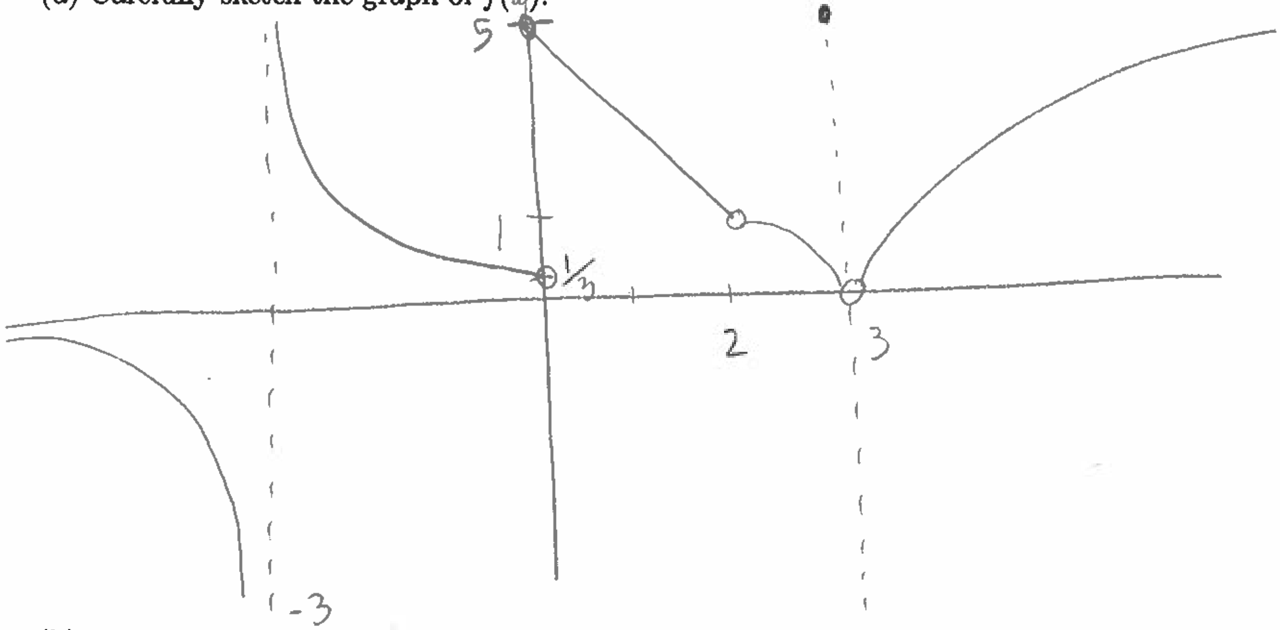
(e) Is $G(x)$ continuous at $x = 5$?

No, $G(x)$ is not continous at $x=5$ b/c $6 = \lim_{x\to 5} G(x) \neq G(5) = -9$.

6. [16 Points] Consider the function defined by

$$f(x) = \begin{cases} \sqrt{x-3} & \text{if } x > 3 \\ 5 & \text{if } x = 3 \\ 1-(x-2)^2 & \text{if } 2 < x < 3 \\ 5-2x & \text{if } 0 \le x < 2 \\ \dfrac{1}{x+3} & \text{if } x < 0 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$.



(b) State the value(s) at which f is discontinuous. Justify your answer(s) using definitions or theorems discussed in class.

(RHL = LHL = 0)

— f discontinuous at $x=3$ b/c although $\lim_{x\to 3} f(x) = 0$ and $f(3) = 5$ is defined, $\lim_{x\to 3} f(x) \neq f(3)$

— f discontinuous at $x=2$ b/c $f(2)$ is undefined

— f discontinuous at $x=0$ b/c $\lim_{x\to 0} f(x)$ DNE b/c RHL $\neq$ LHL

RHL: $\lim_{x\to 0^+} f(x) = 5$

LHL: $\lim_{x\to 0^-} f(x) = \frac{1}{3}$

— f discontinuous at $x=-3$ b/c $\lim_{x\to -3} f(x)$ DNE b/c RHL $\neq$ LHL

RHL: $\lim_{x\to -3^+} f(x) = +\infty$

LHL: $\lim_{x\to -3^-} f(x) = -\infty$

Bonus #1

$$\frac{d}{dx}\left[\frac{x}{y+1}\right] = \frac{d}{dx}\left[x^2 - y^2\right]$$

$$\frac{(y+1)(1) - x \cdot \frac{dy}{dx}}{(y+1)^2} = 2x - 2y\frac{dy}{dx}$$

$$y+1 - x\frac{dy}{dx} = 2x(y+1)^2 - 2y(y+1)^2\frac{dy}{dx}$$

$$2y(y+1)^2\frac{dy}{dx} - x\frac{dy}{dx} = 2x(y+1)^2 - y - 1$$

Isolate + Solve

$$\left(2y(y+1)^2 - x\right)\frac{dy}{dx} = 2x(y+1)^2 - y - 1$$

$$\frac{dy}{dx} = \boxed{\frac{2x(y+1)^2 - y - 1}{2y(y+1)^2 - x}}$$

$$\left.\frac{dy}{dx}\right|_{(1,0)} = \frac{2(1)^2 - 0 - 1}{\underbrace{2(0)(1)^2}_{0} - 1} = \frac{2-1}{-1} = \frac{1}{-1} = -1$$

$$y - 0 = -1(x-1)$$

$$\boxed{y = -x + 1}$$

Bonus #2

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$f(x) = \frac{x^2+1}{7-x^3}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{(x+h)^2+1}{7-(x+h)^3} - \dfrac{x^2+1}{7-x^3}}{h}$$

$$= \lim_{h\to 0} \frac{(x^2+2xh+h^2+1)(7-x^3) - (x^2+1)(7-x^3-3x^2h-3xh^2-h^3)}{(7-(x+h)^3)(7-x^3)} \cdot \frac{1}{h}$$

$$= \lim_{h\to 0} \frac{7x^2+14xh+7-x^5-2x^4h-x^3h^2-x^3-(7x^2-x^5-3x^4h-3x^3h^2-x^2h^3+7-x^3-3x^2h-3xh^2-h^3)}{h(7-(x+h)^3)(7-x^3)}$$

$$= \lim_{h\to 0} \frac{14xh+x^4h-x^3h^2+3x^3h^2+x^2h^3+3x^2h+3xh^2+h^3}{h(7-(x+h)^3)(7-x^3)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(14x+x^4-x^3h+3x^3h+x^2h^2+3x^2+3xh+h^2)}{\cancel{h}(7-(x+h)^3)(7-x^3)}$$

$$= \boxed{\frac{x^4+3x^2+14x}{(7-x^3)^2}}$$

Match Next Page : Quotient Rule

Check:

Quotient Rule

$$\frac{d}{dx}\left(\frac{x^2+1}{7-x^3}\right) = \frac{(7-x^3)(2x)-(x^2+1)(-3x^2)}{(7-x^3)^2}$$

$$= \frac{14x - 2x^4 + 3x^4 + 3x^2}{(7-x^3)^2}$$

$$= \boxed{\frac{x^4+3x^2+14x}{(7-x^3)^2}}$$