Summer Academy Practice Exam #1 Anser Key 2019

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

• Please *show* all of your work and *justify* all of your answers.

1. Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $\lim_{x\to-7} \dfrac{x^2+5x-14}{x^2-4x+4} \overset{\text{DSP}}{=} \dfrac{0}{81} = \boxed{0}$

(b) $\lim_{x\to-6} \dfrac{f(x^2)+5x-8}{[f(x)]^2+5x+14} =$ where $f(x)=x+2$

$$\lim_{x\to-6} \frac{f(x^2)+5x-8}{[f(x)]^2+5x+14} \overset{\frac{0}{0}}{=} \lim_{x\to-6} \frac{x^2+2+5x-8}{[x+2]^2+5x+14} = \lim_{x\to-6} \frac{x^2+5x-6}{x^2+4x+4+5x+14}$$

$$= \lim_{x\to-6} \frac{x^2+5x-6}{x^2+9x+18} = \lim_{x\to-6} \frac{(x+6)(x-1)}{(x+6)(x+3)} = \lim_{x\to-6} \frac{x-1}{x+3} = \frac{-7}{-3} = \boxed{\frac{7}{3}}$$

(c) $\lim_{x\to8} \dfrac{3-\sqrt{x+1}}{x^2-7x-8} \overset{\frac{0}{0}}{=} \lim_{x\to8} \dfrac{3-\sqrt{x+1}}{x^2-7x-8}\cdot\left(\dfrac{3+\sqrt{x+1}}{3+\sqrt{x+1}}\right) = \lim_{x\to8} \dfrac{9-(x+1)}{(x^2-7x-8)(3+\sqrt{x+1})}$

$$= \lim_{x\to8} \frac{8-x}{(x-8)(x+1)(3+\sqrt{x+1})} = \lim_{x\to8} \frac{-(x-8)}{(x-8)(x+1)(3+\sqrt{x+1})}$$

$$= \lim_{x\to8} \frac{-1}{(x+1)(3+\sqrt{x+1})} = \frac{-1}{(9)(3+\sqrt{9})} = \frac{-1}{(9)(6)} = \boxed{\frac{-1}{54}}$$

(d) $\lim_{x\to4} \dfrac{\dfrac{3-x}{x-5}-\dfrac{3}{7-x}}{x^2-x-12} \overset{\frac{0}{0}}{=} \lim_{x\to4} \dfrac{\left(\dfrac{(3-x)(7-x)-3(x-5)}{(x-5)(7-x)}\right)}{x^2-x-12} = \lim_{x\to4} \dfrac{\left(\dfrac{21-10x+x^2-3x+15}{(x-5)(7-x)}\right)}{x^2-x-12} =$

$$\lim_{x\to4} \frac{\left(\dfrac{x^2-13x+36}{(x-5)(7-x)}\right)}{x^2-x-12}$$

$$= \lim_{x\to4} \frac{x^2-13x+36}{(x-5)(7-x)}\cdot\left(\frac{1}{x^2-x-12}\right) = \lim_{x\to4} \frac{(x-9)(x-4)}{(x-5)(7-x)}\cdot\left(\frac{1}{(x-4)(x+3)}\right)$$

$$= \lim_{x\to4} \frac{x-9}{(x-5)(7-x)(x+3)} = \frac{4-9}{(4-5)(7-4)(4+3)} = \frac{-5}{(-1)(3)(7)} = \boxed{\frac{5}{21}}$$

(e) $\lim_{x\to7} \dfrac{x^2-4x-21}{x^2-3x} \overset{\text{DSP}}{=} \dfrac{0}{28} = \boxed{0}$ (Trust this answer)

(f) $\displaystyle\lim_{x\to 4} \frac{x^2-3x-4}{|x-4|} = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\displaystyle\lim_{x\to 4^+} \frac{x^2-3x-4}{|x-4|} = \lim_{x\to 4^+} \frac{x^2-3x-4}{x-4} = \lim_{x\to 4^+} \frac{(x-4)(x+1)}{x-4} = \lim_{x\to 4^+} x+1 = 5$

LHL: $\displaystyle\lim_{x\to 4^-} \frac{x^2-3x-4}{|x-4|} = \lim_{x\to 4^-} \frac{x^2-3x-4}{-(x-4)} = \lim_{x\to 4^-} \frac{(x-4)(x+1)}{-(x-4)} = \lim_{x\to 4^+} -(x+1) = -5$

Recall $|x-4| = \begin{cases} x-4 & \text{if } x-4 \geq 0 \\ -(x-4) & \text{if } x-4 < 0 \end{cases} = \begin{cases} x-4 & \text{if } x \geq 4 & \longleftarrow \text{RHL} \\ -(x-4) & \text{if } x < 4 & \longleftarrow \text{LHL} \end{cases}$

WARNING: The $|x-4|$ does not just cancel with the $x-4$. You must examine the two cases for the absolute value, because we are approaching 7.

(g) $\displaystyle\lim_{x\to 4} \frac{x^2-2x-8}{x^2-5x+4}^{\left(\frac{0}{0}\right)} = \lim_{x\to 4} \frac{(x-4)(x+2)}{(x-4)(x-1)} = \lim_{x\to 4} \frac{x+2}{x-1} \overset{\text{DSP}}{=} \frac{6}{3} = \boxed{2}$

(h) $\displaystyle\lim_{x\to -5} \frac{\frac{1}{1-x}-\frac{1}{6}}{x^2+3x-10}^{\left(\frac{0}{0}\right)} = \lim_{x\to -5} \frac{\frac{6-(1-x)}{(1-x)6}}{x^2+3x-10} = \lim_{x\to -5} \frac{5+x}{(1-x)6}\cdot\frac{1}{x^2+3x-10}$

$\displaystyle= \lim_{x\to -5} \frac{5+x}{(1-x)6}\cdot\frac{1}{(x+5)(x-2)} = \lim_{x\to -5} \frac{1}{(1-x)6(x-2)} \overset{\text{DSP}}{=} \frac{1}{(6)6(-7)} = \boxed{-\frac{1}{252}}$

(i) $\displaystyle\lim_{x\to 3} \frac{x^2-12x+27}{x^2-9}^{\left(\frac{0}{0}\right)} = \lim_{x\to 3} \frac{(x-9)(x-3)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{x-9}{x+3} \overset{\text{DSP}}{=} \frac{-6}{6} = \boxed{-1}$

(j) $\displaystyle\lim_{x\to 2} \frac{\sqrt{x+7}-3}{x^2-3x+2} = \lim_{x\to 2} \frac{\sqrt{x+7}-3}{x^2-3x+2}\cdot\frac{\sqrt{x+7}+3}{\sqrt{x+7}+3} = \lim_{x\to 2} \frac{(x+7)-9}{(x^2-3x+2)(\sqrt{x+7}+3)}$

$\displaystyle= \lim_{x\to 2} \frac{x-2}{(x-2)(x-1)(\sqrt{x+7}+3)} = \lim_{x\to 2} \frac{1}{(x-1)(\sqrt{x+7}+3)} \overset{\text{DSP}}{=} \frac{1}{\sqrt{9}+3)} = \frac{1}{\sqrt{9}+3} = \boxed{\frac{1}{6}}$

2. Suppose that $f(x) = \dfrac{x+7}{x-3}$. Compute the difference quotient $\dfrac{f(x+h)-f(x)}{h}$. Simplify your answer until the h in the denominator cancels.

$$\frac{f(x+h)-f(x)}{h} = \frac{\dfrac{(x+h)+7}{(x+h)-3} - \dfrac{x+7}{x-3}}{h} = \frac{\left(\dfrac{(x+h+7)(x-3)-(x+7)(x+h-3)}{(x+h-3)(x-3)}\right)}{h}$$

$$= \frac{x^2+xh+7x-3x-3h-21-(x^2+xh-3x+7x+7h-21)}{h(x+h-3)(x-3)}$$

$$= \frac{x^2+xh+7x-3x-3h-21-x^2-xh+3x-7x-7h+21}{h(x+h-3)(x-3)}$$

$$= \frac{-3h-7h}{h(x+h-3)(x-3)} = \frac{-10h}{h(x+h-3)(x-3)} = \boxed{\frac{-10}{(x+h-3)(x-3)}}$$

3. Consider the two functions $f(x) = \dfrac{1+x}{1-x}$ and $g(x) = \dfrac{1}{x}$.

(a) Compute $f \circ g(x)$. Simplify your answer to a single fraction. State the Domain.

$$f \circ g(x) = f(g(x)) = f\left(\frac{1}{x}\right) = \frac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}} = \frac{\dfrac{x}{x}+\dfrac{1}{x}}{\dfrac{x}{x}-\dfrac{1}{x}} = \frac{\dfrac{x+1}{x}}{\dfrac{x-1}{x}} = \frac{x+1}{x}\cdot\left(\frac{x}{x-1}\right) = \boxed{\frac{x+1}{x-1}}$$

Domain $f \circ g$: $\{x | x \neq 0, 1\}$. Recall, for x to be in the domain of $f \circ g$, it must be in the domain of g first, and THEN the output $g(x)$ must be in the domain of f.

(b) Compute $g \circ f(x)$. Simplify your answer to a single fraction. State the Domain.

$$g \circ f(x) = g(f(x)) = g\left(\frac{1+x}{1-x}\right) = \frac{1}{\dfrac{1+x}{1-x}} = \boxed{\frac{1-x}{1+x}}$$

Domain $g \circ f$: $\{x | x \neq -1, 1\}$

(c) Compute $f \circ f(x)$. Simplify your answer to a single fraction. State the Domain.

$$f \circ f(x) = f(f(x)) = f\left(\frac{1+x}{1-x}\right) = \frac{1+\dfrac{1+x}{1-x}}{1-\dfrac{1+x}{1-x}} = \frac{\dfrac{1-x}{1-x}+\dfrac{1+x}{1-x}}{\dfrac{1-x}{1-x}-\dfrac{1+x}{1-x}} = \frac{\dfrac{1-x+1+x}{1-x}}{\dfrac{1-x-(1+x)}{1-x}}$$

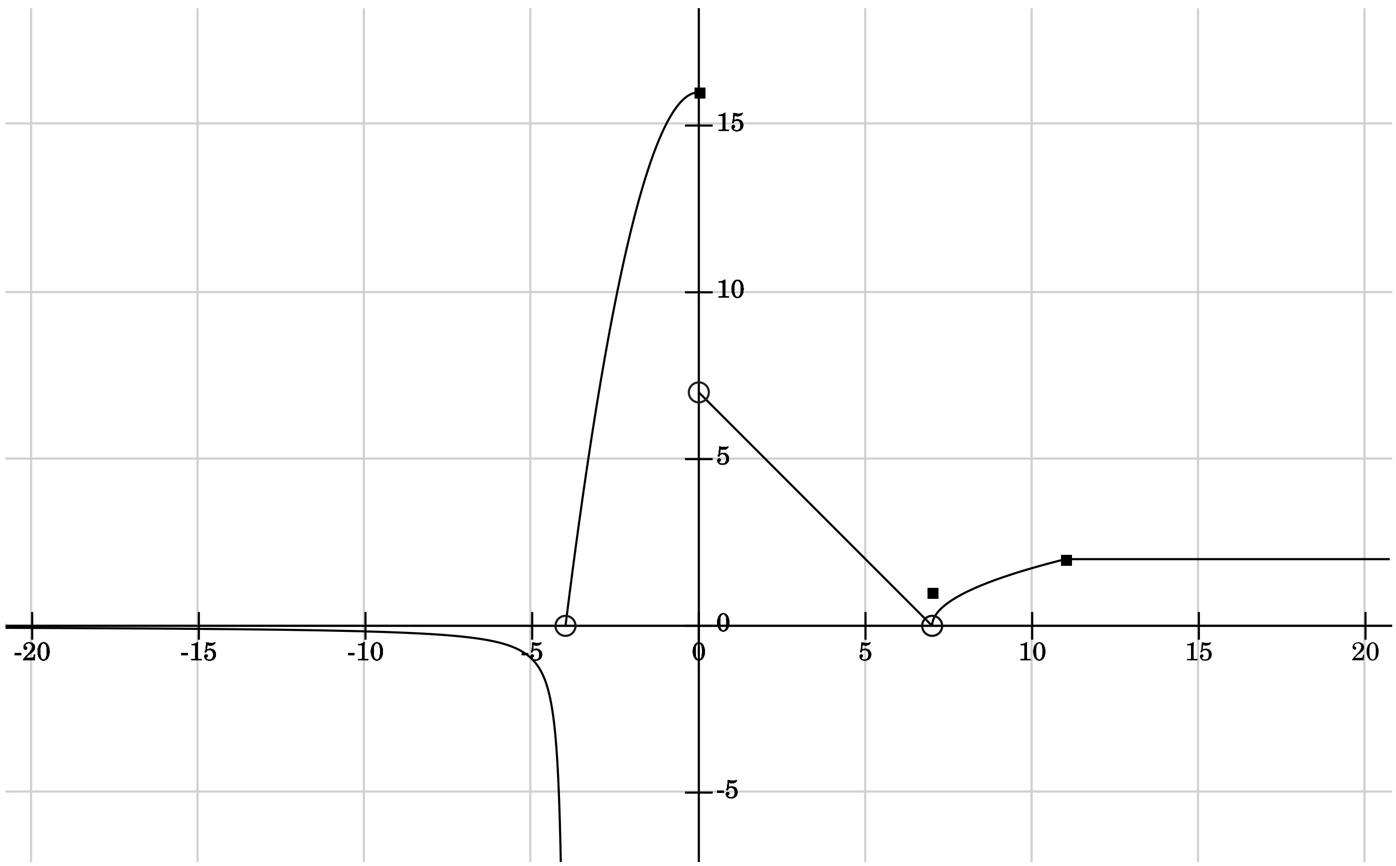
$$= \frac{\dfrac{1-x+1+x}{1-x}}{\dfrac{1-x-1-x)}{1-x}} = \frac{\dfrac{2}{1-x}}{\dfrac{-2x}{1-x}} = \frac{2}{1-x}\cdot\left(\frac{1-x}{-2x}\right) = \boxed{-\frac{1}{x}}$$

Domain $f \circ f$: $\{x | x \neq 0, 1\}$

4. Consider the function defined by

$$f(x) = \begin{cases} 2 & \text{if } x \geq 11 \\ \sqrt{x-7} & \text{if } 7 < x < 11 \\ 1 & \text{if } x = 7 \\ 7 - x & \text{if } 0 < x < 7 \\ 16 - x^2 & \text{if } -4 < x \leq 0 \\ \dfrac{1}{x+4} & \text{if } x < -4 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$.



(b) State the Domain of the function $f(x)$.

Domain= $\boxed{\{x : x \neq -4\}}$

(c) Compute $\lim_{x \to -4} f(x) =$ $\boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

RHL: $\lim_{x \to -4^+} f(x) = \lim_{x \to -4^+} 16 - x^2 = 0$

LHL: $\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} \frac{1}{x+4} = \frac{1}{0^-} = -\infty$

(d) Compute $\lim_{x \to 0} f(x) =$ $\boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}}$

RHL: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 7 - x = 7$

LHL: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 16 - x^2 = 16$

(e) Compute $\lim_{x \to 7} f(x) = \boxed{0}$ since RHL=LHL

RHL: $\lim_{x \to 7^+} f(x) = \lim_{x \to 7^+} \sqrt{x-7} = 0$

LHL: $\lim_{x \to 7^-} f(x) = \lim_{x \to 7^-} 7 - x = 0$

(f) Compute $\lim_{x \to 11} f(x) = \boxed{2}$

RHL: $\lim_{x \to 11^+} f(x) = \lim_{x \to 11^+} 2 = 2$

LHL: $\lim_{x \to 11^-} f(x) = \lim_{x \to 11^-} \sqrt{x-7} = \sqrt{4} = 2$

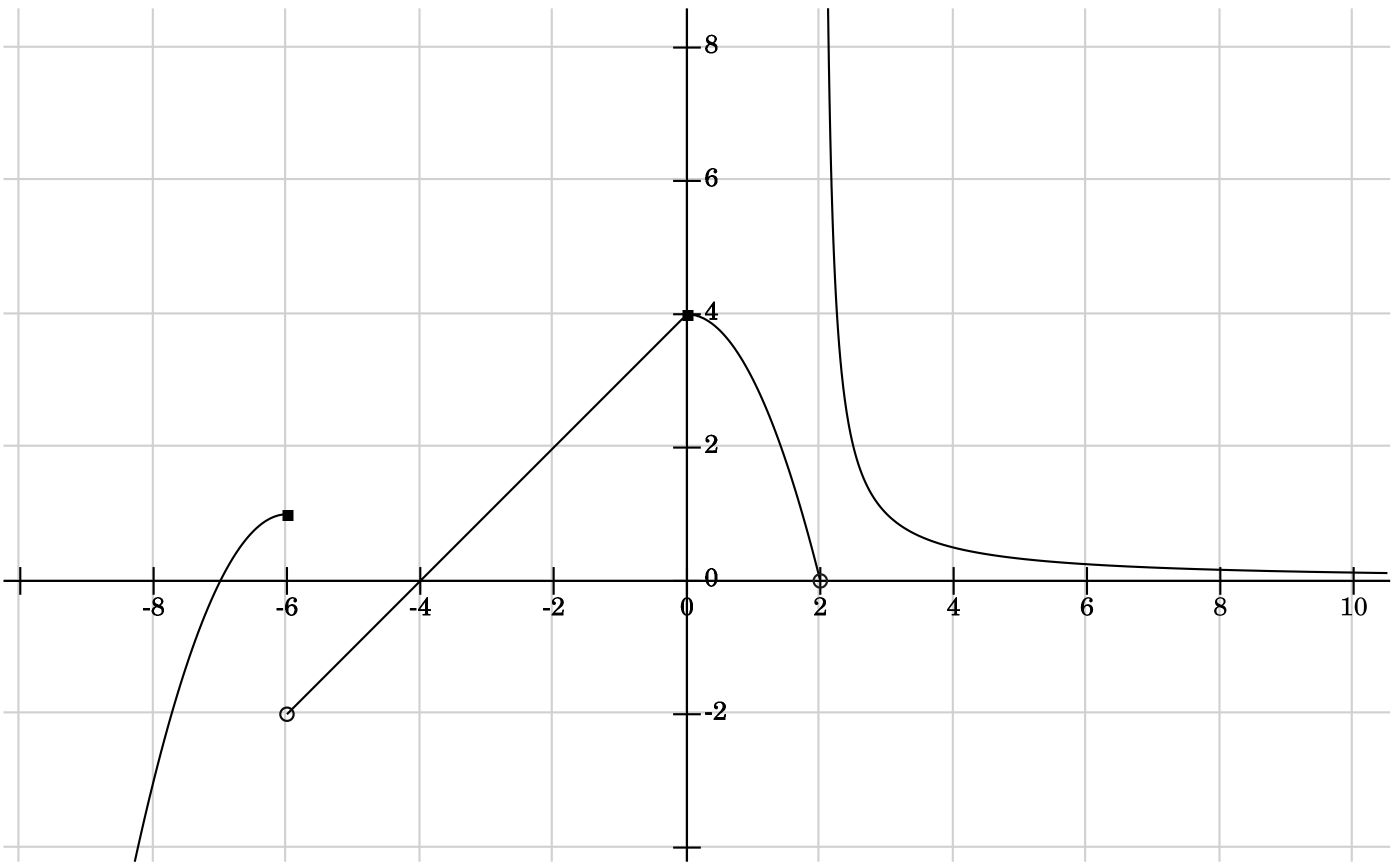
(g) Compute $\lim_{x \to \infty} f(x) = \boxed{2}$

(h) Compute $\lim_{x \to -\infty} f(x) = \boxed{0}$

5. Consider the function defined by

$$f(x) = \begin{cases} \dfrac{1}{x-2} & \text{if } x > 2 \\ 4 - x^2 & \text{if } 0 \le x < 2 \\ x + 4 & \text{if } -6 < x < 0 \\ 1 - (x+6)^2 & \text{if } x \le -6 \end{cases}$$

Graph $f(x)$.



Answer the following questions. Justify your answers.

(a) $\lim_{x \to -6} f(x) =$ $\boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x \to -6^+} f(x) = -2$

LHL: $\lim_{x \to -6^-} f(x) = 1$

(b) $\lim_{x \to 0} f(x) = \boxed{4}$ b/c RHL=LHL

RHL: $\lim_{x \to 0^+} f(x) = 4$

LHL: $\lim_{x \to 0^-} f(x) = 4$

(c) $\lim_{x \to 2} f(x) = \boxed{\text{DNE b/c RHL} \neq \text{LHL}}$

RHL: $\lim_{x \to 2^+} f(x) = +\infty$

LHL: $\lim_{x \to 2^-} f(x) = 0$

(d) $\lim_{x \to \infty} f(x) = \boxed{0}$

(e) $\lim_{x \to -\infty} f(x) = \boxed{-\infty}$