Summer Academy   Midterm Exam #1   June 30, 2019

• This is a closed-book examination. No books, notes, calculators, cell phones, communication devices of any sort, webpages, or other aids are permitted.

• Simplify your answers.

• Please *show* all of your work and *justify* all of your answers. (You may use the backs of pages for additional work space.)

1. [36 Points] Evaluate each of the following limits. Please **justify** your answers. Be clear if the limit equals a value, $+\infty$ or $-\infty$, or Does Not Exist.

(a) $$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3}\cdot\left(\frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}\right) = \lim_{x\to 3} \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)}$$

$$= \lim_{x\to 3} \frac{x-3}{(x-3)(\sqrt{x+1}+2)} = \lim_{x\to 3} \frac{1}{\sqrt{x+1}+2} \overset{\text{L.L.}}{=} \frac{1}{2+2} = \boxed{\frac{1}{4}}$$

(b) $$\lim_{x\to 5} \frac{x^2-3x-10}{|x-5|} = \boxed{\text{DOES NOT EXIST}}, \text{ RHL}\neq\text{LHL}$$

RHL: $$\lim_{x\to 5^+} \frac{x^2-3x-10}{|x-5|} \overset{\frac{0}{0}}{=} \lim_{x\to 5^+} \frac{x^2-3x-10}{x-5} = \lim_{x\to 5^+} \frac{(x-5)(x+2)}{x-5} = \lim_{x\to 5^+} x+2 \overset{\text{DSP}}{=} \boxed{7}$$

LHL: $$\lim_{x\to 5^-} \frac{x^2-3x-10}{|x-5|} \overset{\frac{0}{0}}{=} \lim_{x\to 5^-} \frac{x^2-3x-10}{-(x-5)} = \lim_{x\to 5^-} \frac{(x-5)(x+2)}{-(x-5)} = \lim_{x\to 5^-} -(x+2) \overset{\text{DSP}}{=} \boxed{-7}$$

Here, recall that $$|x-5| = \begin{cases} x-5 & \text{if } x-5\geq 0 \\ -(x-5) & \text{if } x-5<0 \end{cases} = \begin{cases} x-5 & \text{if } x\geq 5 \leftarrow \text{RHL case} \\ -(x-5) & \text{if } x<5 \leftarrow \text{LHL case} \end{cases}$$

(c) $$\lim_{x\to 1} \frac{x^2-5x+3}{x^2-2x} = \frac{1-5+3}{1-2} \overset{\text{DSP}}{=} \frac{-1}{-1} = \boxed{1}$$

(d) $$\lim_{x\to 3} \frac{\frac{2}{x-1}-\frac{3}{x}}{x-3} = \lim_{x\to 3} \frac{\frac{2x-3(x-1)}{x(x-1)}}{x-3} = \lim_{x\to 3} \frac{\frac{2x-3x+3}{x(x-1)}}{x-3} = \lim_{x\to 3} \frac{-x+3}{x(x-1)}\left(\frac{1}{x-3}\right)$$

$$= \lim_{x\to 3} \frac{-(x-3)}{x(x-1)}\left(\frac{1}{x-3}\right) = \lim_{x\to 3} \frac{-1}{x(x-1)} \overset{\text{DSP}}{=} \frac{-1}{2(3)} = \boxed{-\frac{1}{6}}$$

(e) $$\lim_{x\to -3} \frac{G(x^2)-x-9}{G(x+6)+x^2+x-6} = \qquad \text{where} \quad G(x) = x-3.$$

$$\lim_{x\to -3}\frac{G(x^2)-x-9}{G(x+6)+x^2+x-6}=\lim_{x\to -3}\frac{x^2-3-x-9}{(x+6)-3+x^2+x-6}$$

$$=\lim_{x\to -3}\frac{x^2-x-12}{x^2+2x-3}=\lim_{x\to -3}\frac{(x-4)(x+3)}{(x-1)(x+3)}=\lim_{x\to -3}\frac{x-4}{x-1}\overset{\text{DSP}}{=}\frac{-7}{-4}=\boxed{\frac{7}{4}}$$

(f) $$\lim_{x\to 2}\frac{x^2-7x+10}{x^2+2x-8}=\lim_{x\to 2}\frac{(x-5)(x-2)}{(x+4)(x-2)}=\lim_{x\to 2}\frac{x-5}{x+4}=-\frac{3}{6}=\boxed{-\frac{1}{2}}$$

2. [8 Points] Suppose that $f(x)=\dfrac{1}{x^2}$. Compute the difference quotient $\dfrac{f(x+h)-f(x)}{h}$. Simplify your answer until the h in the denominator cancels.

$$\frac{f(x+h)-f(x)}{h}=\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}=\frac{\frac{x^2-(x+h)^2}{(x+h)^2x^2}}{h}$$

$$=\frac{\frac{x^2-(x^2+2xh+h^2)}{(x+h)^2x^2}}{h}=\frac{\frac{x^2-x^2-2xh-h^2)}{(x+h)^2x^2}}{h}=\frac{\frac{-2xh-h^2}{(x+h)^2x^2}}{h}$$

$$=\frac{\frac{h(-2x-h)}{(x+h)^2x^2}}{h}=\frac{h(-2x-h)}{(x+h)^2x^2}\cdot\frac{1}{h}$$

$$=\boxed{\frac{-2x-h}{(x+h)^2x^2}}$$

3. [16 Points] Consider the two functions $f(x)=\dfrac{1}{x}$ and $g(x)=x-5$. Compute each of the following. Simplify your answers. **THEN** Sketch each graph.

(a) $f(x+3)=\boxed{\dfrac{1}{x+3}}$

(b) $f(x^2)+3=\boxed{\dfrac{1}{x^2}+3}$

(c) $g(x^2)=\boxed{x^2-5}$

(d) $[g(x)]^2=\boxed{(x-5)^2}$

(e) $g\circ f(x)=g(f(x))=g\left(\dfrac{1}{x}\right)=\boxed{\dfrac{1}{x}-5}$

(f) $f \circ f(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\left(\frac{1}{x}\right)} = \boxed{x}$

(g) $f \circ g(x) = f(g(x)) = f(x-5) = \boxed{\frac{1}{x-5}}$

(h) $g \circ g(x) = g(g(x)) = g(x-5) = (x-5) - 5 = \boxed{x - 10}$

4. [10 Points] For each of the following problems below, sketch any graph for the function f with the description given.

(a) Sketch a graph of any function f for which $\boxed{\lim_{x \to 2} f(x) \textbf{ Exists}}$.

(b) Sketch a graph of any function f for which $\boxed{\lim_{x \to 2} f(x) = 5}$.

(c) Sketch a graph of any function f for which $\boxed{\lim_{x \to 2} f(x) \quad \textbf{Does not Exist}}$.

(d) Sketch a graph of any function f for which $\boxed{\lim_{x \to 2} f(x) = -1}$ and $\boxed{f(2) = 4}$.

(e) Sketch a graph of any function f for which $\boxed{\lim_{x \to 2} f(x) = -\infty}$ and $\boxed{f(2) \text{ is undefined}}$.

5. [20 Points] Consider the function defined by

$$f(x) = \begin{cases} \dfrac{1}{x-3} & \text{if } x > 3 \\ x^2 + 1 & \text{if } 0 < x < 3 \\ -2 & \text{if } x = 0 \\ x + 1 & \text{if } -2 < x < 0 \\ 5 - (x+2)^2 & \text{if } x < -2 \end{cases}$$

(a) Carefully sketch the graph of $f(x)$.

(b) State the **Domain** of the function $f(x)$.

D=$\{x|x \neq -2, 3\}$

(c) Compute
$$\begin{cases} \lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} x+1 = \boxed{-1} \\ \lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} 5-(x+2)^2 = \boxed{5} \\ \lim_{x\to -2} f(x) = \lim_{x\to -2} f(x) \ \boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}} \end{cases}$$

(d) Compute
$$\begin{cases} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2+1 = \boxed{1} \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x+1 = \boxed{1} \\ \lim_{x\to 0} f(x) = \boxed{1} \ \text{RHL=LHL} \end{cases}$$

(e) Compute
$$\begin{cases} \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{1}{x-3} = \frac{1}{0^+} = \boxed{+\infty} \\ \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2+1 = \boxed{10} \\ \lim_{x\to 3} f(x) \ \boxed{\text{DOES NOT EXIST since RHL} \neq \text{LHL}} \end{cases}$$

(f) Compute $\lim_{x\to\infty} f(x) = \boxed{0}$

(g) Compute $\lim_{x\to -\infty} f(x) = \boxed{-\infty}$